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THE ACCOUNTANCY OF INVESTMENT

BY

CHARLES EZRA SPRAGUE, A.M., PH.D., LITT.D., C.P.A.

Late Professor of Accountancy in New York University;
Late President of Union Dime Savings Bank, New York City

With which are incorporated "Logarithms to 12 Places and Their Use in Interest Calculations" and "Amortization" by the same author

REVISED BY

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J. E. STERRETT
ROBERT H. MONTGOMERY

PREFACE

Among the published works of the late Colonel Sprague, there are four which deal particularly with certain mathematical phases of accounting, viz.: "Text Book of the Accountancy of Investment"; "Amortization"; "Logarithms to 12 Places and Their Use in Interest Calculations"; and "Extended Bond Tables." Since the author's death in March, 1912, it has become desirable to combine the first three of these publications into one volume, in order to serve more effectively the needs of business men and students of accounting by presenting the material in compact and convenient form.

The present volume is the result of this consolidation. In it has been incorporated everything of practical value contained in the three works mentioned, while at the same time the special features of those books have been amplified by additional text matter and problems, wherever such additions have seemed desirable for the sake of more adequate treatment.

In conformity with the usual and commendable practice of Colonel Sprague, the reviser has avoided as far as possible the use of the more difficult mathematical demonstrations and formulas, believing that thereby the book will prove of greater utility to practicing accountants, bankers, and other business men. On this point we quote from the author's original preface: "Treatises on this subject (Mathematics of Investment), written for actuarial students, are invariably too difficult, except for those who have not only been highly trained in algebra, but are fresh in its use, and this makes the subject forbidding to many minds. I have

made all my demonstrations arithmetical and illustrative, but, I think, none the less convincing and intelligible."

It is believed that for a considerable number of readers, the tables of logarithms given in Part III will prove of great utility in those cases where more than ordinary accuracy is required, and where special tables are, at times, imperative. To quote again from the preface of Colonel Sprague: "Rough results will answer for approximative purposes; but where it is desirable, for instance, to construct a table of amortization, sinking fund, or valuation of a lease at an unusual rate, for a large amount and for a great many years, exactness is desirable and becomes self-proving at the end."

A whole book is required for the ordinary tables of logarithms of six or seven places, while the tables here presented are contained in a few pages and give accurate results to twelve places of decimals. The processes with these tables are necessarily somewhat slower than with those of six or seven places, but their use is fully justified where greater accuracy in results is desirable.

The tables of compound interest, present worth, annuities, and sinking funds, carried to eight places of decimals, have been retained in this edition. Such tables are practically indispensable in securing accurate computations. The index of subjects at the end of the book is a new feature and will facilitate quick reference to any information desired.

LEROY L. PERRINE.

New York City, January, 1914.

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EXPLANATION OF SYMBOLS

For the sake of brevity and clearness, certain constantly recurring expressions have been represented in the text by symbols. The following list comprises all of those which are not self-explanatory.

1 = \$1, £1, or any other unit of value.

a = the amount of \$1 for a given time at a given rate.

A = the amount of an annuity of \$1 for a given time at a given rate.

c = the cash, or coupon, rate of interest (or the cash payment) for a single period.

d = the rate of discount for a single period.

D = the discount on \$1 for a given time at a given rate.

i = the rate of interest (or the income) for a single period.

I = the compound interest on \$1 for a given time at a given rate..

j = the effective rate of interest for one year.

n = an indefinite number of units.

p = the present worth of \$1 for a given time at a given rate.

P = the present worth of an annuity of \$1 for a given time at a given rate.

$r = (1 + i)$, the periodic ratio of increase.

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THE ACCOUNTANCY OF INVESTMENT

Part I—The Mathematics of Investment

CHAPTER I

CAPITAL AND REVENUE

§ 1. Definition of Capital

That portion of wealth which is set aside for the production of additional wealth is capital. The capital of a business, therefore, is the whole or a part of the assets of the business, and of course appears on the active or debit side of its balance sheet. This is the sense in which the word "capital" is used in economics; but in bookkeeping the term "Capital account" is often used in quite another sense to mean accounts on the credit or passive side, which denote proprietorship. To prevent confusion, the use of the expression "Capital account" will be avoided.

§ 2. The Use of Capital

In active business, capital must be employed, and, in order to produce more wealth, it must be combined with skill and industry. Businesses, and consequently their accounting methods, vary as to the manner in which capital is used. Cash is convertible into potential capital of any kind desired. In a manufacturing business it is exchanged

for machinery, appliances, raw materials, and labor which transforms these raw materials into finished products. In a mercantile business cash is expended for goods, bought at one price to sell at another, and for collecting, displaying, caring for, advertising, and delivering goods. To bridge over the time between selling and collecting, additional capital is required, usually known as "working capital," but which might more appropriately be styled "waiting capital." Thus we may analyze each kind of business, and show that the nature of its capital assets depends on the character of the business.

§ 3. Sources of Capital

On the credit side of the balance sheet the capital must be accounted for in such a manner as to show its sources. Here there are two sharply divided classes: *loan*-capital, or liability, and *own*-capital, or proprietorship. The great distinction is that the latter participates in the profits and bears the losses, while the former takes its share irrespective of the success of the concern. It is the own-capital which is referred to in the phrase "Capital account."

§ 4. Investment

While we often speak of a man's capital as being invested in a business, we use the word "investment" more strictly when we confine it to the non-participating sense. Thus we say, "He not only owns a business, but he has some investments besides." In the strictest sense, then, investment implies divesting one's self of the possession and control of one's assets, and granting such possession and control to another. The advantage of the use of capital must be great enough to enable the user to earn more than the sum which he pays to the investor, or capitalist. There are many cases where the surrender is not absolute, and

more or less risk is assumed by the investor. This is not absolute investment, but to some extent partnership. The essence of strict investment is the vicarious earning of a share in gains which do not depend on the business skill of the investor.

§ 5. Revenue

All investments are made with a view to obtaining revenue, which is the share of the earnings given for the use of capital. Revenue takes three forms: interest, rent, and dividends—the first two corresponding to strict investment, and the latter to participation.

§ 6. Interest and Rent

These do not essentially differ. Both are stipulated payments for the use of capital; but in case of rent the identical physical asset received by the lessee must be returned by him on the completion of the contract. If you borrow a dollar, you may repay any dollar you please; if you hire a house or a horse, you may not return any house or any horse, but must produce the identical one you had. Interest and rent are both proportionate to time.

§ 7. Dividends

These are profits paid over to the owners of the own-capital, whether partners or shareholders. The amount is subtracted from the collective assets and paid over to the separate owners. Theoretically there is neither profit nor loss in this distribution. I have more cash, but my share in the collective assets is exactly that much less. The cash is distributed partly because it is needed by the participants for consumption; and partly because no more capital can be profitably used in the enterprise. Some concerns, however, such as banks, which can profitably use more capital and

whose shareholders do not require cash for consumption, frequently refrain from dividing the periodical profits, or distribute but a small portion of them.

The accumulation of the profits, however, inures just as surely to the benefit of the shareholders, and is usually realizable through increased value of the shares upon sale. Thus, dividends are not strictly revenue, though the shareholder may treat them as such; his dividend may be so regular as practically to be fixed, or his shares may be preferential, so that to some extent he is receiving an ascertained amount; or, as in case of a leased railway, the dividend may be expressly stipulated in a contract. Still, legally speaking, the dividend is instantaneous, and does not accrue, like interest and rent.

§ 8. Laws of Interest

As all investments are really purchases of revenue, and as the value of an investment depends largely upon the amount of revenue derivable therefrom, and as the typical form of revenue is interest, it is necessary to study the laws of interest, including those more complex forms—annuities, sinking funds, and amortization. Although there is a special branch of accountancy—the actuarial—which deals not only with these subjects, but with life and other contingencies, it is yet very necessary for the general accountant to understand at least their fundamental principles.

CHAPTER II

INTEREST

§ 9. Interest

As ordinarily defined, interest is "money paid for the use of money." A better definition from a mathematical standpoint would be, "the increase of indebtedness through lapse of time." Since the production of additional wealth is dependent on the processes of nature, and since these processes require time, it is equitable that compensation for an increase in time should be made by an increase in indebtedness. The "money paid" of the first definition is a payment on account of the general debt (including interest); the direct effect of interest is to increase the debt, while the direct effect of a payment is to reduce it.

§ 10. Essentials of Interest Contract

The contract, express or implied, regarding an interest transaction, must take into consideration the following items:

(1) Principal. The number of units of value (dollars, pounds, francs, marks, etc.) originally loaned or invested.

(2) Rate. The part of the unit of value (usually a small number of hundredths) which is added to each such unit by the lapse of one unit of time.

(3) Frequency. The length of the unit of time, measured in years, months, or days. Weeks are not used as time units, nor are parts of a day.

(4) Time. The number of units of time during which the indebtedness is to continue.

§ 11. Interest Rate

The *rate* is usually spoken of as so much per cent per period, or term. Thus, if the contract provides for the payment of three cents each year for the use of each dollar of principal, the rate may be expressed, .03 per annum, 3 per centum per annum, 3 per cent, or simply 3%. Where the period is not a year, but a smaller unit of time, it is nevertheless customary to speak of the annual rate. For instance, instead of saying, "3% per half-year," we say, "6%, payable semi-annually." In the same way, 1% per quarter would be 4%, payable quarterly. In our discussions of interest, however, we shall treat of *periods,* and of the *rate per period,* in order to avoid confusion. The interest rate will be designated by the small letter i; as, $i = .06$. At the end of the first period the increased indebtedness, corresponding to the original unit of indebtedness at the beginning of the term, is $1 + i$ (1.06), a very important quantity in computation. The subject of rates of interest will be discussed in greater detail in Chapter VIII, "Nominal and Effective Rates."*

§ 12. Principal

Since each dollar increases just as much as every other dollar, the general practice is to consider the principal as *one dollar* and, when the proper interest thereon has been found, to multiply it by the *number* of dollars.

§ 13. Simple and Compound Interest

Interest is assumed to be paid when due. If it is not so paid, it ought to be added to the principal, and interest should be computed on the increased principal. But the law does not directly sanction this compounding of interest,

*For discussion of the causes of higher or lower interest rates, see "The Rate of Interest," by Prof. Irving Fisher.

and *simple interest* is spoken of as if it were a distinct species in which the original principal remains unchanged, even though interest is in default. There is really no such thing as simple interest, since the interest money which is wrongfully withheld by the borrower, may be by him employed, and thus compound interest be earned. But the wrong party gets the benefit of the compounding. All the calculations of finance depend upon *compounding interest,* which is the only rational and consistent method. When there is occasion hereafter to speak of the interest for one period, it will be called "single interest."

§ 14. Punctual Interest

The usual interest contract provides that the increase shall be paid off in cash at the end of each period, restoring the principal to its original amount. Let c denote the cash payment; then $1+i-c=1$; and the second term would repeat the same process. The payment of cash for interest must not be regarded as the interest; it is a cancellation of part of the increased principal. Many persons, and even courts, have been misled by the old definition of interest—"money paid for the use of money"—into treating uncollected or unmatured interest as a nullity, though secured in precisely the same way as the principal.

§ 15. Computation of Interest

But the interest money may not be paid exactly at the end of each term, either in violation of the contract or by a special clause permitting it to run on, or by the debt being assigned to a third party at a price which modifies the true interest rate. In this case the question arises: How shall the interest be computed for the following periods? This gives rise to a distinction between *simple* and *compound* interest.

§ 16. Simple Interest

During the second period, although the borrower has in his hands an increased principal, $1 + i$, he is at simple interest charged with interest only on 1, and has the free use of i, which, though small, has an earning power proportionate to that of 1. His indebtedness at the end of the second term is $1 + 2i$, and thereafter $1 + 3i$, $1 + 4i$, etc. After the first period he is *not* charged with the agreed percentage of the sum actually employed by him, and this to the detriment of the creditor. For any scientific calculation, simple interest is impossible of application.

§ 17. Compound Interest

The indebtedness at the end of the first period is $1 + i$, and up to this point *punctual, simple,* and *compound* interest coincide. But in compound interest the fact is recognized that the increased principal, $1 + i$, is *all* subject to interest during the next period, and that the debt increases by *geometrical* progression, not arithmetical. The increase from 1 to $1 + i$ is regarded, not as an addition of i to 1, but as a multiplication of 1 by the *ratio of increase* $(1 + i)$. We shall designate the ratio of increase by r when convenient, although this is merely an abbreviation of $1 + i$, and the two expressions are at all times interchangeable.

§ 18. Computation of Compound Interest

At the end of the first period (which is equivalent to the beginning of the second period), the actual indebtedness is $1 + i$. This amount is the equitable principal for the second period, and it should be again increased in the ratio $1 + i$. The total indebtedness at the end of the second period (which is equivalent to the beginning of the third period) is therefore $1 \times (1 + i) \times (1 + i)$. For the sake of brevity, this may be written $1 \times (1 + i)^2$, the figure 2 (called an ex-

ponent) indicating that the expression $(1+i)$ is to be taken twice as a factor. Since the expression $(1+i)$ equals the rate, a still simpler way of indicating the indebtedness at the end of the first period is r; at the end of the second period, r^2. At the end of the third period the indebtedness will have become r^3; and at the end of period t, it will have become r^t.

§ 19. Comparison of Simple and Compound Interest

The following schedule shows the accumulations of interest for several periods, giving a comparison between the simple interest computations and the compound interest computations:

Time	Indebtedness Based on Simple Interest	Indebtedness Based on Compound Interest*
Beginning of 1st period..	1	1
Beginning of 2nd period.	$1+i$	$1+i$
Beginning of 3rd period.	$1+2i$	$(1+i)^2$
Beginning of 4th period..	$1+3i$	$(1+i)^3$
Beginning of 5th period..	$1+4i$	$(1+i)^4$
etc.		

*For the benefit of students familiar with algebra, it may be pointed out that $(1+i)^2 = 1+2i+i^2$. This differs from the simple interest computation by the small quantity i^2. Similarly, $(1+i)^3 = 1+3i+3i^2+i^3$, which differs from the simple interest result, $1+3i$, by the quantity $3i^2+i^3$. Tests may be readily made of the computations by substituting a numerical rate, say .06, in place of i. If this be done, the simple interest result at the beginning of the 4th period is found to be 1 + (3 times .06), or 1.18. The compound interest result would be:

1	= 1.
plus 3 times .06	= .18
plus 3 times $.06^2$, or 3 times .0036	= .0108
plus $.06^3$	= .000216
That is, $(1.06)^3$	= 1.191016

§ 20. The Day as a Time Unit

Coming now to a discussion of *frequency* and *time,* in connection with the subject of interest, we find that the smallest unit of time is one day, since the law does not recognize interest for fractions of a day. The legal day begins at midnight and ends on the following midnight. In reckoning from one day to another, the day *from which* should be excluded. Thus, if a loan is made at any hour on the third day of the month and is paid at any hour on the fourth, there is one day's interest due, the interest being for the fourth day and not for the third. Practically it is the nights that count. If five midnights have passed since the loan was made, then the accrued interest is for a period of five days.

§ 21. The Month as a Time Unit

As has been previously stated, weeks are not used as time units. The next longer interest period after a day is a month. Calendar months are computed as follows: Commence at the day from which the reckoning is made, and exclude that day; then the day in the next month having the same number will at its close complete the first month; the second month will end with the same numbered day, and so on to the same day of the final month. A difficulty arises in the case where the initial date is the 31st, while the last month has only thirty days or less. In this case the interest month ends with the last day of the calendar month. For example, one month from January 31st, 1912, was February 29th; one month from January 30th or 29th, in the same year, also terminated on February 29th; in a common year, not a leap year, the last day of a period one month from January 28th, 29th, 30th or 31st, would be February 28th.

§ 22. Half and Quarter Years

Since there are no fractions of a day in interest computations, it becomes necessary to inquire what is meant by a half-year or by a quarter. In the State of New York the Statutory Construction Law (Laws of 1892, Chapter 677, § 25) solves this difficulty by prescribing that a half-year is not $182\frac{1}{2}$ days, but six calendar months; and that a quarter is not $91\frac{1}{4}$ days, but three calendar months.

§ 23. Partial Interest Periods

In practice any fraction of an interest period is computed at the corresponding fraction of the rate, although theoretically this is not quite just. For example, if the interest rate is 6% per annum, payable annually, making the ratio of increase 1.06, then it is customary to consider the ratio of increase for a half-year as 1.03; whereas theoretically it should be the square root of 1.06, or slightly over 1.029563.

If the regular period is one year, any odd days should be reckoned as 365ths of a year. Also, if the contract is for days only and there is no mention of months, quarters, or half-years, then also a day should be regarded as 1/365 of a year. But when the contract is for months, quarters, or half-years, any fractional time should be divided into months, and there is usually an odd number of days left over. In New York, doubt exists as to how these odd days should be treated, whether on a 365-day basis or on a 360-day basis.

Before 1892 there was no doubt. The statute distinctly stated that a number of days less than a month should be estimated for the purpose of interest computations as 30ths of a month, or, consequently, 360ths of a year. This was a most excellent provision, and merely enacted what had been the custom long before. The so-called "360-day" interest tables are based upon this rule. In 1892, however,

the revisers of the statutes of the State of New York dropped this sensible provision and left the question open. No judicial decision has since been rendered on the subject, but many good lawyers think that the odd days should be computed as 365ths of a year. In business nearly every one calls the odd days 360ths, and it is only in legal accountings that there can be any question. It would be well if the old provision could be re-enacted by law or re-established by the courts.

§ 24. Changing the Day Basis

If the interest for a certain number of odd days has been computed on a 360-day basis, a change may be readily made to a 365-day basis by subtracting from such interest 1/73 of itself. On the other hand, if the interest for an odd number of days has been ascertained on a 365-day basis, the addition of 1/72 of itself to this amount will give the interest on a 360-day basis.

§ 25. The Amount—First Period

The principal and interest taken together constitute the amount. At the end of the first half-year period, the amount of $1.00 at 6% interest, payable semi-annually, is $1.03. Instead of considering the $1.00 and the 3 cents as two separate items to be added together, it is best to consider the operation as the single one of multiplying $1.00 by the ratio of increase, 1.03. Sometimes the error is made of considering that the original principal of $1.00 is multiplied by $1.03, or, in other words, that a certain number of dollars is multiplied by another number of dollars. It is well to emphasize, in this connection, the old principle given in arithmetic, that one concrete number cannot be multiplied by another concrete number. We cannot multiply dollars by dollars, or feet by feet, or horses by dollars. The multiplicand may be either a concrete or an abstract number, but the multiplier must always be abstract.

§ 26. The Amount—Subsequent Periods

The principal which is employed during the second period is \$1.03. It is evident that this, like the original \$1.00, should be multiplied by the ratio 1.03. The new amount will be the square of 1.03, which we may write:

$$
\begin{aligned}
&1.03 \times 1.03 \\
\text{or, }\ &1.03^2 \\
\text{or, }\ &1.0609
\end{aligned}
$$

This is the new amount on interest during the third period. At the end of the third period the amount will be:

$$
\begin{aligned}
&1.03 \times 1.03 \times 1.03 \\
\text{or, }\ &1.03^3 \\
\text{or, }\ &1.092727
\end{aligned}
$$

At the end of the fourth period the amount becomes:

$$
\begin{aligned}
&1.03^4 \\
\text{or, }\ &1.12550881
\end{aligned}
$$

Possibly at this point the number of decimal places may be unwieldy. If we desire to have only seven decimal places, we reject the final 1, rounding the result *off* to 1.1255088; if we prefer to use only six places, we round the result *up* to 1.125509, which is more nearly correct than 1.125508.

§ 27. Exponents and Powers

In some of the following paragraphs, it will be necessary to speak occasionally of exponents and powers. In the expression 1.03^2, the figure 2 is called an exponent, and it means (as indicated in the preceding paragraph) that 1.03 is to be taken twice as a factor. In other words, the number is to be multiplied by itself. The result, 1.0609 (which equals 1.03^2), is said to be the second power of 1.03; 1.092727 is the third power of 1.03, and so on. (See also § 38.)

§ 28. Finding the Amount—Compound Interest

The amount of $1.00 at the end of any number of periods is obtained by taking such a *power* of the ratio of increase as is indicated by the number of periods; or, in other words, by multiplying $1.00 by the ratio as many times as there are periods. If the original principal be subtracted from the amount, the remainder is the compound interest. For example, in § 26, the amount of $1.00 at the ratio 1.03, for four periods, is $1.12550881; and the compound interest is $.12550881.

§ 29. Present Worth

The present worth of a future sum is a smaller sum which, put at interest, will amount to the future sum. The present worth of $1.00 is such a sum as, at the given rate and for the given period, will amount to $1.00. In order to illustrate the method of ascertaining the present worth, let us suppose that it is desired to find the present worth of $1.00, due in four years, the ratio of increase being 1.03 per annum. The required figure must evidently be such that, when multiplied four times in succession by 1.03, the result will be $1.00. Therefore, by using the reverse process, division, the required figure may be obtained. The first operation, by ordinary long division, results as follows:

```
1.03 ) 1.00000000 ( .970873
         927
         ---
          730
          721
          ---
            900
            824
            ---
             760
             721
             ---
              390
              309
              ---
               81
```

The result, rounded up at the 6th place, is .970874, this being the present worth of \$1.00 due in one period at 3% interest. The present worth for two periods may be obtained either by again dividing .970874 by 1.03, or by multiplying .970874 by itself, or by dividing 1 by 1.0609 (the square of 1.03), each of which operations gives the same result, .942596. The present worth for three periods may also be obtained in several ways, the result being the same in all cases, .915142, or $\frac{1}{1.03^3}$. The present worth for four periods is $\frac{1}{1.03^4}$, or .888487.

§ 30. Present Worth and Amount Series

If we arrange these results in reverse order, followed by \$1.00 and by the amounts computed in § 26, we have a continuous series:

$$
\begin{aligned}
1 \div 1.03^4 &= .888487 \\
1 \div 1.03^3 &= .915142 \\
1 \div 1.03^2 &= .942596 \\
1 \div 1.03 &= .970874 \\
&\ 1. \\
1 \times 1.03 &= 1.03 \\
1 \times 1.03^2 &= 1.0609 \\
1 \times 1.03^3 &= 1.092727 \\
1 \times 1.03^4 &= 1.125509
\end{aligned}
$$

§ 31. Relation between Present Worth and Amount

In the foregoing series, which might be extended indefinitely upward and downward, every term is a *present worth* of the one which immediately follows it, and an *amount* of the one which immediately precedes it. When one number is the amount of another, the latter number is the present worth of the former. For example, .888487 is the present worth of 1.125509 for 8 interest periods; and, on the other hand, 1.125509 is the amount of .888487, for the same number of interest periods and at the same ratio.

In some instances in this series, a present worth and its corresponding amount are reciprocals (that is, their product is 1), but this is true only when the two figures are distant an equal number of periods from 1, the present-worth figure being upward from 1 and the amount figure being downward from 1. Thus, .915142, or $1 \div 1.03^3$, is the reciprocal of 1.092727, or 1.03^3.

§ 32. Formation of Series

If any term of the series be multiplied by 1.03, the product will be the next following term; if it be divided by 1.03 or (which amounts to the same thing) be multiplied by .970874, the result will be the next preceding term. Since multiplying by 1.03 is easier than dividing by it, and also easier than multiplying by .970874, the easiest way of obtaining the different numbers in the series is to compute first the smallest number (in this case, .888487), and then perform successive multiplications by 1.03. A brief process for finding this initial number will be explained in the next chapter.

§ 33. Discount

In considering the present worth of $1.00 for a single period (.970874), it is evident that the original $1.00 has been diminished by .029126, which is a little less than .03; in fact it is $.03 \div 1.03$. This difference, .029126, is called the *discount*. In the present worth for two periods, the discount is $1 - .942596$, or .057404. This discount for two periods, and likewise the discounts for three or more periods, are called *compound discounts*.

§ 34. Computing Compound Discount

The compound discount for any number of periods may be found either by subtracting the present worth from 1, or

by finding the present worth of the compound interest for the same time and at the same rate. As an illustration, suppose that it is desired to find the compound discount of \$1.00 for three periods at 3%. First, we may subtract the present worth, .915142, from 1, which gives the compound discount as .084858. Second, we may divide the compound interest (.092727) by the amount of \$1.00 for the three periods (1.092727), which gives the compound discount the same as before, .084858.

§ 35. Formulas for Interest Calculations

We may reduce the rules to more compact form by the use of symbols. Let a represent the amount of \$1.00 for any number of periods (n periods); p the present worth; i the rate of interest per period; d the rate of discount per period, and n the number of periods. Let the compound interest be represented by I, and the compound discount by D. Then, by § 17, the ratio of increase is $(1+i)$. By § 28, $a=(1+i)^n$; and $\mathrm{I}=a-1$. By § 29, $p=1\div$ a or $(1+i)^n$; and by § 34, $\mathrm{D}=1-p$, or $\mathrm{I}\div a$.

§ 36. Use of Logarithms

The method of ascertaining the values of a and p through successive multiplications and divisions for a large number of periods, is intolerably slow. A much briefer way is by the use of certain auxiliary numbers called logarithms as explained in the next chapter.

CHAPTER III

THE USE OF LOGARITHMS

§ 37. Purpose of Logarithms

For multiplying or dividing a great many times by the same number, or for finding powers and roots, there is no device superior to a table of *logarithms*. Although the computation of logarithms—as in the formation of a table of logarithms—requires a knowledge of algebra, the practical *use* of logarithmic tables does not require such knowledge. The aid derived from such tables is purely arithmetical, and the occasional prejudice against logarithms as something mysterious or occult is without reasonable foundation.

§ 38. Exponents, Powers, and Roots

We have seen in § 27 that an exponent is a number written at the right and slightly above another number to indicate how many times the latter is to be taken as a factor; and also that a power is the result obtained by taking any given number a certain number of times as a factor. We now add that a root is the number repeated as a factor to form a power. The following table exemplifies roots, exponents, and powers:

Roots and Exponents	Powers
$2^2 = 2 \times 2$	$= 4$
$3^2 = 3 \times 3$	$= 9$
$4^2 = 4 \times 4$	$= 16$
etc.	
$2^3 = 2 \times 2 \times 2$	$= 8$
$3^3 = 3 \times 3 \times 3$	$= 27$
$4^3 = 4 \times 4 \times 4$	$= 64$
etc.	

The root of a number is called its first power. When the root is taken twice as a factor, the result is called the second power, or square; when taken three times, the result is the third power, or cube; we may in like manner obtain the fourth, fifth, or any power of a root by repeating it as a factor the required number of times.

§ 39. Logarithms as Exponents

Now, logarithms are merely exponents of certain roots which are called *bases*. The common system of logarithms is based upon the number 10, this number being the basis of our decimal system of numeration.

Taking a specific illustration, let us multiply six 10's together, $10 \times 10 \times 10 \times 10 \times 10 \times 10$; we may write the result as:

1,000,000
or, 10^6
or, the sixth power of ten.

The small figure "6" is the exponent of the power. A series of some of the powers of 10 might be represented as follows:

1,000,000	or	10^6
100,000	"	10^5
10,000	"	10^4
1,000	"	10^3
100	"	10^2
10	"	10^1
1	"	10^0

From the above series, the following observations may be made:

(1) The number of zeroes in any number in the first column is the same as the exponent in the second column.

(2) Each term in the first column is *one-tenth* of the

one above it, while in the second column each exponent is *one less* than the exponent above it. This leads to the result that $10^0 = 1$, which at first seems impossible. It is difficult to understand how 10, taken zero times as a factor, equals 1, but such nevertheless is a fact, as can be easily demonstrated by algebra.*

(3) By adding any two exponents in the second column, we may find the result of multiplying together the two corresponding numbers in the first column. For example, 10^2 (or 100) times 10^3 (or 1,000) equals 10^{2+3}; i.e., 10^5 (or 100,000); in other words, by adding the logarithms of two numbers we obtain the logarithm of their product. Again, by finding the difference between any two logarithms in the second column, we may find the quotients of the corresponding numbers in the first column. For example, $10^{5-2} = 10^3 = 1{,}000$, which $= 100{,}000 \div 100$; i.e., by subtracting logarithms the logarithms of quotients are found.

Suppose we should wish to obtain the second power of 10^3; the exponent (or index) of the second power is 2; and $10^{3 \times 2} = 10^6 = 1{,}000 \times 1{,}000$, or 1,000,000; from which it appears that by multiplying the logarithm 3 by the index 2 we have obtained the square of 10^3, or 1,000. Again $10^{6 \div 2} = 10^3 = 1{,}000$, or the square root of 1,000,000; from which it appears that, by dividing the logarithm 6 by the index 2, we obtain the square root of 10^6.

§ 40. Rules and Symbols of Logarithms

Summarized very briefly, the rules of logarithms, deduced from the foregoing illustrations, are as follows:

---

* By the use of the equation:

$$\frac{x^2}{x^2} = 1; \quad \frac{x^2}{x^2} = x^{2-2} = x^0$$

$$\begin{aligned} \text{Therefore, } x^0 &= 1 \\ \text{or, if } x &= 10 \\ 10^0 &= 1 \end{aligned}$$

(1) By *adding* logarithms, numbers are *multiplied.*

(2) By *subtracting* logarithms, numbers are *divided.*

(3) By *multiplying* logarithms, numbers are raised to *powers.*

(4) By *dividing* logarithms, the *roots* of numbers are extracted.

The last two of these rules are the only ones necessary to be employed in the calculations of compound interest.

With this preliminary explanation of logarithms, the series in § 39 may be rewritten and "extended"; the symbol *nl* meaning "the number whose logarithm is."

The base being 10,
1,000,000 is the number whose logarithm is 6,
or, in contracted form,

$10^6 =$	1,000,000.	*nl*	6
$10^5 =$	100,000.	*nl*	5
$10^4 =$	10,000.	*nl*	4
$10^3 =$	1,000.	*nl*	3
$10^2 =$	100.	*nl*	2
$10^1 =$	10.	*nl*	1
$10^0 =$	1.	*nl*	0
	.1	*nl*	—1
	.01	*nl*	—2
	.001	*nl*	—3
	.0001	*nl*	—4
	.00001	*nl*	—5

Occasionally the symbol *ln* will also be used, its meaning being "the logarithm of the number."

From the above, it will be seen that the logarithm of a number is merely the exponent which indicates the power to which some given number, called the base, would have to be raised in order to give that number. If the number

were 100 and the base 10, then 10 would have to be raised to the *second* power to give 100; in other words, the logarithm of 100, with 10 as a base, is 2. If the number were 100,000, the base still being 10, then 10 would have to be raised to the *fifth* power to give the number; or we may say, in different language, that the logarithm of 100,000, with 10 as a base, is 5.

§ 41. The Two Parts of a Logarithm

The logarithms of a few numbers have already been given, but for practical use in calculating we need the logarithms of a great many others. From the series in § 39, it may be readily inferred that the numbers between 1 and 10 must have their logarithms between 0 and 1; that is, the logarithms of these numbers must be fractions. In tables of logarithms, these fractions are expressed as decimals, the usual number of decimal places being seven. Similarly, the numbers between 10 and 100 have their logarithms between 1 and 2; that is, these logarithms are 1 plus a decimal fraction.

To give a few illustrations:

The	logarithm	of	10 = 1.
"	"	"	12 = 1.0792
"	"	"	20 = 1.3010
"	"	"	50 = 1.6990
"	"	"	90 = 1.9542
"	"	"	99 = 1.9956
"	"	"	100 = 2.

It will be observed that there are usually two parts to a logarithm—the decimal part and the whole number preceding the decimal. The decimal part is known as the *mantissa*, and the whole number as the *characteristic*. From the mantissa of a logarithm, we are able (through the aid of loga-

rithmic tables) to determine the corresponding number, except as to the position of its decimal point. This latter is determined by the characteristic, which, for numbers greater than 1, is always one less than the number of places to the left of the decimal point. For numbers less than 1, the characteristic is said to be negative, and it is equal to the number of places to the right from the decimal point to the place occupied by the first significant figure of the decimal. A negative characteristic is indicated by a short dash placed above it. A characteristic may thus be either positive or negative, but a mantissa is always positive.

§ 42. Mantissa Not Affected by Position of the Decimal Point

In the logarithms of 20, 200, 2,000, 20,000, 200,000, 2,000,000, etc., we shall find the same decimal part, .301 030 (which is the logarithm of 2), preceded by the figures 1, 2, 3, 4, 5, 6, etc. This same thing is true of any combination of figures; that is to say, whatever may be the position of the decimal point in a number, the logarithm of that number always has the same decimal fraction, or mantissa.

Thus, if the logarithm of 2.378 is .376 212, then,

.0002378	*nl*	$\bar{4}.376\ 212$
.002378	*nl*	$\bar{3}.376\ 212$
.02378	*nl*	$\bar{2}.376\ 212$
.2378	*nl*	$\bar{1}.376\ 212$
2.378	*nl*	.376 212
23.78	*nl*	1.376 212
237.8	*nl*	2.376 212
2,378.	*nl*	3.376 212
23,780.	*nl*	4.376 212
237,800.	*nl*	5.376 212
etc.		etc.

§ 43. Four-Place Table of Logarithms

Illustrations will now be given of the properties of logarithms, and for this purpose a table of the logarithms of numbers from 10 to 99, inclusive, to four places of decimals, is given on the following pages. This is a very simple table of logarithms, and is known as a four-place table.

The ordinary tables of logarithms are calculated to seven places of decimals. If it is desired to multiply the number 82 by 1.03 fifty times in succession, ordinary logarithm tables would give only the first seven figures of the answer. If this operation were performed accurately by the simple processes of multiplication, the answer would contain 103 figures, 3 in front of the decimal point and 100 after it. Since the figures after the first seven are for most purposes insignificant, the result obtained by logarithms will be near enough even if rounded off at the sixth figure.

N	0	1	2	3	4	5	6	7	8	9
10	0000	043	086	128	170	212	253	294	334	374
11	0414	453	492	531	569	607	645	682	719	755
12	0792	828	864	899	934	969	*004	*038	*072	*106
13	1139	173	206	239	271	303	335	367	399	430
14	1461	492	523	553	584	614	644	673	703	732
15	1761	790	818	847	875	903	931	959	987	*014
16	2041	068	095	122	148	175	201	227	253	279
17	2304	330	355	380	405	430	455	480	504	529
18	2553	577	601	625	648	672	695	718	742	765
19	2788	810	833	856	878	900	923	945	967	989
20	3010	032	054	075	096	118	139	160	181	201
21	3222	243	263	284	304	324	345	365	385	404
22	3424	444	464	483	502	522	541	560	579	598
23	3617	636	655	674	692	711	729	747	766	784
24	3802	820	838	856	874	892	909	927	945	962
25	3979	997	*014	*031	*048	*065	*082	*099	*116	*133
26	4150	166	183	200	216	232	249	265	281	298
27	4314	330	346	362	378	393	409	425	440	456
28	4472	487	502	518	533	548	564	579	594	609
29	4624	639	654	669	683	698	713	728	742	757

* See explanation at the end of this section.

N	0	1	2	3	4	5	6	7	8	9
30	4771	786	800	814	829	843	857	871	886	900
31	4914	928	942	955	969	983	997	*011	*024	*038
32	5051	065	079	092	105	119	132	145	159	172
33	5185	198	211	224	237	250	263	276	289	302
34	5315	328	340	353	366	378	391	403	416	428
35	5441	453	465	478	490	502	514	527	539	551
36	5563	575	587	599	611	623	635	647	658	670
37	5682	694	705	717	729	740	752	763	775	786
38	5798	809	821	832	843	855	866	877	888	899
39	5911	922	933	944	955	966	977	988	999	*010
40	6021	031	042	053	064	075	085	096	107	117
41	6128	138	149	160	170	180	191	201	212	222
42	6232	243	253	263	274	284	294	304	314	325
43	6335	345	355	365	375	385	395	405	415	425
44	6435	444	454	464	474	484	493	503	513	522
45	6532	542	551	561	571	580	590	599	609	618
46	6628	637	646	656	665	675	684	693	702	712
47	6721	730	739	749	758	767	776	785	794	803
48	6812	821	830	839	848	857	866	875	884	893
49	6902	911	920	928	937	946	955	964	972	981
50	6990	998	*007	*016	*024	*033	*042	*050	*059	*067
51	7076	084	093	101	110	118	126	135	143	152
52	7160	168	177	185	193	202	210	218	226	235
53	7243	251	259	267	275	282	292	300	308	316
54	7324	332	340	348	356	364	372	380	388	396
55	7404	412	419	427	435	443	451	459	466	474
56	7482	490	497	505	513	520	528	536	543	551
57	7559	566	574	582	589	597	604	612	619	627
58	7634	642	649	657	664	672	679	686	694	701
59	7709	716	723	731	738	745	752	760	767	774
60	7782	789	796	803	810	818	825	832	839	846
61	7853	860	868	875	882	889	896	903	910	917
62	7924	931	938	945	952	959	966	973	980	987
63	7993	*000	*007	*014	*021	*028	*035	*041	*048	*055
64	8062	069	075	082	089	096	102	109	116	122

* See explanation at the end of this section.

N	0	1	2	3	4	5	6	7	8	9
65	8129	136	142	149	156	162	169	176	182	189
66	8195	202	209	215	222	228	235	241	248	254
67	8261	267	274	280	287	293	299	306	312	319
68	8325	331	338	344	351	357	363	370	376	382
69	8388	395	401	407	414	420	426	432	439	445
70	8451	457	463	470	476	482	488	494	500	506
71	8513	519	525	531	537	543	549	555	561	567
72	8573	579	585	591	597	603	609	615	621	627
73	8633	639	645	651	657	663	669	675	681	686
74	8692	698	704	710	716	722	727	733	739	745
75	8751	756	762	768	774	779	785	791	797	802
76	8808	814	820	825	831	837	842	848	854	859
77	8865	871	876	882	887	893	899	904	910	915
78	8921	927	932	938	943	949	954	960	965	971
79	8976	982	987	993	998	*004	*009	*015	*020	*025
80	9031	036	042	047	053	058	063	069	074	079
81	9085	090	096	101	106	112	117	122	128	133
82	9138	143	149	154	159	165	170	175	180	186
83	9191	196	201	206	212	217	222	227	232	238
84	9243	248	253	258	263	269	274	279	284	289
85	9294	299	304	309	315	320	325	330	335	340
86	9345	350	355	360	365	370	375	380	385	390
87	9395	400	405	410	415	420	425	430	435	440
88	9445	450	455	460	465	469	474	479	484	489
89	9494	499	504	509	513	518	523	528	533	538
90	9542	547	552	557	562	566	571	576	581	586
91	9590	595	600	605	609	614	619	624	628	633
92	9638	643	647	652	657	661	666	671	675	680
93	9685	689	694	699	703	708	713	717	722	727
94	9731	736	741	745	750	754	759	763	768	773
95	9777	782	786	791	795	800	805	809	814	818
96	9823	827	832	836	841	845	850	854	859	863
97	9868	872	877	881	886	890	894	899	903	908
98	9912	917	921	926	930	934	939	943	948	952
99	9956	961	965	969	974	978	983	987	991	996

* See explanation at the end of this section.

In the table preceding, the figures in the column headed "N" denote the numbers whose logarithms are given. These numbers must be considered in conjunction with the numbers at the top of the remaining ten columns; in other words, we can find the logarithm not only of the number (say) 34, but also of 34.1, 34.2, 34.3, etc., and similarly of .34, .341, .342, .343, etc., and of 3,400, 3,410, 3,420, 3,430, etc.

The figures in the columns headed "0," "1," "2," etc., represent simply the decimal parts (or mantissas) of the logarithms; the whole (or integral) parts of the logarithms must always be determined by inspection (§ 41).

The column headed "0" has four places of figures, while the following columns have only three places. This is done to save space, since a fourth figure is assumed to be prefixed, this figure being the same as the first figure in the four-place column. There is an exception to this rule in the case of figures prefixed by an asterisk, and three or four examples will serve to make this clear:

Number	Corresponding Logarithm
2.50	.3979
2.51	.3997
2.52	.4014 (*not* .3014)
2.53	.4031 (*not* .3031)
etc.	etc.

§ 44. Multiplication by Logarithms

As stated in § 40, there are four general rules regarding logarithms, and these will now be illustrated in order.

Rule 1: The sum of the logarithms of two or more numbers is the logarithm of their product.

	2	*nl*	.3010
	3	*nl*	.4771
2×3	6	*nl*	.7782
	4	*nl*	.6021
	14	*nl*	1.1461
4×14	56	*nl*	1.7482
	5	*nl*	.6990
	20	*nl*	1.3010
5×20	100	*nl*	2.0000

In these and other illustrations, there may be apparent errors in the final decimal figure, due to throwing away or adding on parts of decimals, as the case might be. For example, in the first illustration given above, the logarithm of 2 to six places is .301030, and the logarithm of 3 is .477121; the logarithm of 6, the product, is .778151.

§ 45. Division by Logarithms

The logarithm of a product is obtained by finding the sum of the logarithms of the factors, and, as division is the converse of multiplication, the logarithm of a quotient is obtained by finding the difference between the logarithms of the dividend and divisor.

Rule 2: The difference of the logarithms of two numbers is the logarithm of their quotient.

Required the quotient of $6 \div 2$.

6	*nl*	.7782		
2	*nl*	.3010		
$6 \div 2$	*nl*	.4771	*ln*	3

Required the quotient of $42 \div 14$.

42	*nl*	1.6232		
14	*nl*	1.1461		
$42 \div 14$	*nl*	.4771	*ln*	3

Required the quotient of $100 \div 4$.

100	*nl*	2.0000		
4	*nl*	.6021		
$100 \div 4$	*nl*	1.3979	*ln*	25

§ 46. Powers by Logarithms

Rule 3: The logarithm of the power of a number is equal to the logarithm of the number multiplied by the exponent of the power.

Let it be required to find the third power of 2, that is, 2^3, or the product of $2 \times 2 \times 2$.

2	*nl*	.3010		
2^3	*nl*	$3 \times .3010$		
		or, .9030	*ln*	8

Required the fourth power of 5, that is, 5^4.

5	*nl*	.6990		
5^4	*nl*	$4 \times .6990$		
		or, 2.7960	*ln*	625

It has been observed in this connection (§ 38), that the second power is usually called the *square,* and the third power the *cube.*

§ 47. Roots by Logarithms

The fourth general rule regarding logarithms refers to the extraction of roots. We have seen in § 38 that, if a certain number is a power of another, we call the latter number a *root* of the former. For example, since $2 \times 2 \times 2 \times 2 \times 2 = 32$, it is said that 32 is the 5th power of 2, and that the 5th root of 32 is 2. The usual way of expressing this latter fact is:

$$\sqrt[5]{32} = 2$$

$$\text{or, } 32^{1/5} = 2$$

With the above explanation, the fourth rule is now stated, and it will be observed that it is the converse of the third rule.

Rule 4: The logarithm of the root of a number is equal to the logarithm of the number divided by the index of the root.

As an illustration, let it be required to find the square root of 49.

$$
\begin{array}{llll}
 & 49 & nl & 1.6902 \\
\sqrt{49}, & \text{or } 49^{1/2} & nl & \tfrac{1}{2} \text{ of } 1.6902 \\
 & & & \text{or, } .8451 \quad ln \quad 7
\end{array}
$$

Required the cube root of 512.

$$
\begin{array}{llll}
 & 512 & nl & 2.7093 \\
\sqrt[3]{512}, & \text{or } 512^{1/3} & nl & \tfrac{1}{3} \text{ of } 2.7093 \\
 & & & \text{or, } .9031 \quad ln \quad 8
\end{array}
$$

§ 48. Fractional Exponents

Such an exponent as $\frac{3}{4}$ requires explanation. It signifies the third power of the fourth root, or the fourth root of the third power. Thus, the value of $10^{3/4}$ may be ascertained by finding the fourth root of 10, and then getting the cube of this root; or by finding the cube of 10, which is 1,000, and then taking the fourth root (or the square root of the square root) of 1,000. By the methods of arithmetic, the value of $10^{3/4}$ is thus found to be 5.62+; or, in other words, the logarithm of 5.62 is approximately .75. It is interesting to compare this result with the table in § 43, where it is indicated that the logarithm of 5.62 is .7497, which is very close to .7500. Fractional exponents may be expressed as decimal, instead of common, fractions; and, in fact, that is what most logarithms are: simply fractional exponents of 10, expressed decimally.

§ 49. Use of Logarithms in Computing Compound Interest

To demonstrate the use of logarithms in compound interest, let us take an example and work it out, illustrating each step. We will take 3% as the rate, the same as already used (§§ 25-30), but endeavor to find the amount for 50 periods, instead of for 4 periods.

The ratio of increase is 1.03. Looking for the logarithm (to eight decimal places) of this ratio (Chambers' or Babbage's tables, page 192) we find this line:

No.	0	1	2	3	4	5	6	7	8	9
10300	0128 3722	4144	4566	4987	5409	5831	6252	6674	7096	7517

The meaning of this line is that the logarithms are as follows:

1.03	*nl*	.01283722
1.03001	*nl*	.01284144
1.03002	*nl*	.01284566
1.03003	*nl*	.01284987
....		....
1.03009	*nl*	.01287517

The first figures of both the numbers and the logarithms are given only once in the table, which saves space in printing and time in searching.

Since 1.03 is to be taken as a factor 50 times, we must multiply its logarithm by 50, as stated in Rule 3 (§ 46). This gives:

$$50 \times .01283722 = .6418610$$

The result is the logarithm of the answer; for, when we have found the corresponding number, we shall know the value of 1.03^{50}.

We must now look in the right-hand columns for the logarithm figures .6418610. We first look for the 641, which stands out by itself, overhanging a blank space

(Chambers' or Babbage's tables, page 73), and we find that the nearest approach to .6418610 is .6418606, which latter is indicated as the logarithm of the number 4.3839. The next nearest logarithm is .6418705, which corresponds to the number 4.3840. The following tabulation shows the details more clearly:

Logarithm	Corresponding Number
.6418606	4.3839
.6418610	To be determined
.6418705	4.3840

It is evident that the number to be determined lies between 4.3839 and 4.3840, which differ by .0001. The difference between the first and third logarithms is .0000099, and between the first and second logarithms is .0000004. For practical purposes, we take 4/99 of the difference between the numbers (.0001), and add this amount to the smaller number, thus obtaining the required number 4.383904. In order to assist in determining the decimal value of 4/99 and similar fractions, little difference-tables are usually given in the margins of the pages of logarithm tables, the table for 99 reading as follows:

	99
1	10
2	20
3	30
4	40
5	50
6	59
7	69
8	79
9	89

The meaning of this table is that $40/99 = .40$; $4/99 =$

.04; 8/99 = .079; etc. By the use of these small tables, the labor of dividing is thus avoided.

§ 50. Accuracy of Logarithmic Results

The amount of $1.00 compounded for 50 periods at 3% is seen to be $4.383904. The result is slightly inaccurate in the last figure, for the reason that two decimal places were lost by multiplying. Had we taken the ten-figure logarithm

on page XVIII of Chambers' tables.......	.0128372247
this multiplied by 50 would give..........	.641861235
or, rounded off at the 7th place...........	.6418612
which gives the more accurate result......	4.383906

§ 51. Logarithms to Fifteen Places

Since it is necessary, for problems involving many periods, to use a very extended logarithm, there is given in Part III of the present volume, tables of fifteen-place logarithms for a number of different ratios of increase $(1+i)$. These are at much closer intervals than any table previously published, and, with a ten-figure book of logarithms, will give exact results to the nearest cent on $1,000,000.00.

§ 52. Use of Logarithms in Present Worth Calculations

We will further exemplify the advantage of the logarithmic method by solving a present worth problem. Let it be required to find the present worth of $1.00 due in 50 periods, compounded at 3% per period. Multiplying the logarithm of 1.03 by 50, just as in § 50, we obtain .641861235. But it is the reciprocal of 1.03^{50}, or $1 \div 1.03^{50}$, which we wish to obtain; hence we must subtract .641861235 from the logarithm of 1, which is 0.

$$
\begin{array}{rr}
 & 0.000000000 \\
 & \underline{0.641861235} \\
\text{Remainder,} & \bar{1}.358138765
\end{array}
$$

In subtracting a greater from a less logarithm, we get a negative whole number (as shown by the minus above), the decimal part being positive and obtained by ordinary subtraction.

Neglecting the $\bar{1}$, for the moment, we search in the right-hand column for .358138765, and find that .3581253 is the logarithm of 2.2810; and proceeding as in § 50, we find that .3581388 is the logarithm of 2.281071. The decimal point, however, must be moved one place to the left, as directed by the characteristic $\bar{1}$; thus giving as the final result, .2281071.

By means of multiplication, we may check the results shown in this and the foregoing sections.

By § 50, 1.03^{50} is.......	4.383906
As above, $1 \div 1.03^{50}$ is...	.2281071

Since these two results are reciprocals, their product should equal unity, or 1. The result of the multiplication is 1.0000000843326, which verifies the accuracy of the previous computations.

CHAPTER IV

AMOUNT OF AN ANNUITY

§ 53. Evaluation of a Series of Payments

We have now investigated the two fundamental problems in compound interest, viz.: to find the amount of a present worth, and to find the present worth of an amount. The next question is a more complex one: to find the amount and the present worth of a series of payments. If these payments are irregular as to period, value, and rate of interest, the only way of finding the amount or the present worth of the series is to make as many separate computations as there are payments, and then find the sum of the results obtained. But, if the payments, periods, and rates of interest are uniform, we can devise a method for finding by one operation the amount or present worth of the whole series.

§ 54. Annuities

A series of payments of like amounts, made at regular periods, is called an annuity; the period does not necessarily need to be a year, but may be a half-year, a quarter, or any other length of time. Thus, if an agreement is made providing for the following payments:

September	9, 1914......	$100.00
March	9, 1915......	100.00
September	9, 1915......	100.00
March	9, 1916......	100.00

there would be an annuity of \$200.00 per annum, payable semi-annually; or, in other words, an annuity of \$100.00 for each half-year period, terminating after four periods. Assuming the rate of interest to be 6% per annum, payable semi-annually (3% per period), let us suppose that it is required to find the total amount to which the annuity will have accumulated on March 9, 1916, and the present worth, on March 9, 1914, of this series of future payments. It is evident that the answer to the first question will be greater than \$400.00, and that the answer to the second question, as shown in the next chapter, will be less than \$400.00.

§ 55. Amount of Annuity

It is easy, in this case, to find the separate amounts of the payments, since the number of terms is very small and since we may avail ourselves of the computations in § 30. A schedule could be made as follows:

Date of Payment	Amount at March 9, 1916
March 9, 1916	\$100.00
September 9, 1915	103.00
March 9, 1915	106.09
September 9, 1914	109.2727
Total,	\$418.3627

§ 56. Calculation of Annuity Amounts

If, however, there were 50 terms instead of 4, the work of computing these 50 separate amounts, by the use of logarithms, or by the shorter process (in this case) of simple multiplication, would be very tedious. To shorten the process let us make up three columns of amounts for four periods, the first being amounts of \$1.00, the second being amounts of \$1.03, and the third being amounts of \$.03. The figures in the second column will accordingly be 1.03 times the

corresponding figures in the first column, while the figures in the third column will be the difference between the corresponding figures in the first two columns.

(1) Amounts of \$1.00	(2) Amounts of \$1.03	(3) Amounts of \$.03
\$1.00	\$1.03	\$.03
1.03	1.0609	.0309
1.0609	1.092727	.031827
1.092727	1.12550881	.03278181
Total, \$4.183627	\$4.30913581	\$.12550881

§ 57. Formation of Tables

We may take the difference between the totals of columns (1) and (2) without actually finding these totals. It will be observed that the first three items in column (2) are the same as the last three items of column (1). The difference between the totals of the two columns, therefore, is the same as the difference between the last item of (2) and the first item of (1); that is, \$1.12550881 less \$1.00, or \$.12550881. This latter figure equals the total of column (3).

§ 58. Use of Tables

It is evident that an annuity of three cents will amount, under the conditions assumed, to twelve cents and the decimal .550881. Accordingly, an annuity of one cent will amount to one-third of \$.12550881, or \$.04183627. An annuity of \$1.00 will amount to 100 times as much, or \$4.183627, while an annuity of \$100.00 will amount to \$418.3627, which agrees exactly with the result obtained by addition, in § 55.

§ 59. Compound Interest as a Base for Annuity Calculations

The amount \$.12550881 (obtained by subtracting \$1.00 from \$1.12550881) is the *compound interest* on \$1.00 for the given rate and time, and the amount \$.03 is the *single interest.* The compound interest on \$1.00, compounded semi-annually at 6%, up to any time corresponds with the amount of an annuity of three cents, payable on exactly the same plan. The amount of the annuity of \$1.00 is \$.12550881 ÷ .03, or \$4.183627; and from this we formulate the rule given in the following section.

§ 60. Rule and Formula for Finding Amount

To find the amount of an annuity of \$1.00 for a given time and at a given rate, divide the compound interest for the total number of periods, by the single interest for one period, both expressed decimally.

To express the rule in a formula, let A represent the amount, not of a single \$1.00, but of an annuity of \$1.00; then $A = I \div i$.

§ 61. Operation of Rule

To illustrate, let us take the case worked out in § 50, where we found the amount of a single dollar at 3%, for 50 periods, to be

periods, to be	\$4.383906
Subtracting one dollar	1.000000
The compound interest is	\$3.383906
Divide this by .03 and we have	\$112.79687

which is the amount to which 50 payments of \$1.00 each, at 3% per period, would accumulate.

CHAPTER V

PRESENT WORTH OF AN ANNUITY

§ 62. Method of Calculation

To find the present worth of an annuity, we can, of course, find the present worth of each payment, and then, by addition, find the total present worth of all the payments; but it will save much labor if we compute the total in one operation, as we computed the amount, and a similar course of reasoning will lead to the desired result.

§ 63. Tables of Present Worth

In the second column of the following table is shown the present worth of $1.00 for 4, 3, 2 and 1 period, respectively, at 3% per period; and in the third and fourth columns are shown similar values of $1.03 and $.03, respectively.

(1) Number of Periods	(2) Present Worths of $1.00	(3) Present Worths of $1.03	(4) Present Worths of $.03
4	$.888487	$.915142	$.026655
3	.915142	.942596	.027454
2	.942596	.970874	.028278
1	.970874	1.000000	.029126
Total,	$3.717099	$3.828612	$.111513

§ 64. Short Method for Finding Present Worth of an Annuity

Since the last three items in column (2) are the same as the first three items in column (3), it is evident that, in order to obtain the difference between the totals of columns (2) and (3), it is not necessary to make the actual additions of these columns, but merely to find the difference between the items not found in both columns. These items are only two, viz., \$.888487 in the second column, and \$1.000000 in the third column. Their difference is \$.111513, which agrees with the total found by the addition of column (4).

§ 65. Present Worth Obtained

The difference between the \$.888487 of the second column and \$1.000000 of the third column, amounting to \$.111513, is the compound discount of \$1.00 for four periods at 3%. When this difference is divided by the single interest (.03), we obtain \$3.71710, which is the same result (rounded up) as that obtained by adding column (2). From this observation, we construct the rule given in the following section:

§ 66. Rule for Present Worth

To find the present worth of an annuity of \$1.00 for a given time at a given rate, divide the compound discount for that time and rate by the single interest for one period, both expressed decimally.

§ 67. Formulas for Present Worth

In symbols, the rule may be expressed, $P = D \div i$. Since, by § 35, $D = I \div a$, we obtain $P = I \div a \div i$, or $P = I \div i \div a$. And since, by § 60, $A = I \div i$, there comes the resulting symbolic rule, $P = A \div a$, the latter part of this equation signifying the present worth of the *amount* of the

annuity. Summarizing, therefore, we have the two symbolic rules:

$$P = D \div i$$
$$P = A \div a$$

§ 68. Analysis of Annuity Payments

It may assist in acquiring a clear idea of the working of an annuity, if an analysis is given of a series of annuity payments from the point of view of the purchaser. For this purpose we will suppose that a person investing $3.7171 at 3%, in an annuity of $1.00 per period, payable at the end of each period, expects to receive at each payment, besides 3% on his principal to date, a portion of that principal, and thus to have his entire principal gradually repaid.

His original principal is........................		$3.7171
At the end of the first period, he receives:		
3% on $3.7171....................	$.1115	
Payment on principal.............	.8885	.8885
Total	$1.0000	
Leaving new principal (which is equivalent to the present worth at three periods)		$2.8286
At the end of the second period, he receives:		
3% on $2.8286..................	$.0849	
Payment on principal.............	.9151	.9151
Total	$1.0000	
Leaving new principal........................		$1.9135
At the end of the third period, he receives:		
3% on $1.9135..................	$.0574	
Payment on principal.............	.9426	.9426
Total	$1.0000	
Leaving new principal........................		$.9709

At the end of the last period, he receives:

3% on \$.9709	\$.0291	
Payment on principal in full	.9709	.9709
Total	\$1.0000	

In the above manner we find that the annuitant has received interest in full on the principal outstanding, and has also received the entire original principal. The correctness of the basis on which we have been working is thus corroborated.

§ 69. Components of Annuity Instalments

It is usual to form a schedule showing the components of each instalment in tabular form:

Date	Total Payment	Payments of Interest	Payments on Principal	Principal Outstanding
March 9, 1914.				\$3.7171
September 9, 1914.	\$1.00	\$.1115	\$.8885	2.8286
March 9, 1915.	1.00	.0849	.9151	1.9135
September 9, 1915.	1.00	.0574	.9426	0.9709
March 9, 1916.	1.00	.0291	.9709	0.0000
	\$4.00	\$.2829	\$3.7171	

Had the purchaser reinvested each instalment at 3%, he would have, at the end, \$4.1836 (§ 55), which is equivalent to his original investment compounded ($\$3.7171 \times 1.1255 = \$4.1836$).

§ 70. Amortization

The payments on principal are known as amortization, which may be defined as the gradual repayment of a principal sum through the resultant operation of two opposing forces —periodical payments and compound interest. The effect

of the periodical payments is to reduce the principal sum, while the effect of the compound interest is to increase it. In ordinary compound interest, each new principal is greater than the preceding principal; while in the case of amortization, each principal is less than the preceding one.

§ 71. Amortization and Present Worth

It will be noticed, from § 69, that each payment on principal, or amortization, for one period, is the present worth of the instalment at the beginning of its period. For example, at the end of the first period, September 9, 1914, a payment on principal is made amounting to $.8885, which is the present worth of the instalment paid on that date ($1.00) for four periods at 3%. From this fact, it follows that, if we know the amount of the instalment, the rate, and the number of remaining periods, we can calculate the amortization included in the instalment.

§ 72. Development of a Series of Amortizations

It will also be noticed that each amortization multiplied by 1.03 becomes the next following, these being a series of present worths; and that thus they may be derived from one another, upwards or downwards.

§ 73. Evaluation by Logarithms

In § 52, by the use of logarithms, we found the present worth of $1.00 for 50 periods, at 3%, to be.. $.2281071
Subtracting this from.................... 1.0000000

we have the compound discount.............. $.7718929
Dividing this by .03, we have.............. $25.72976+
which is the present worth of an annuity of $1.00 for 50 periods, at 3%. Thus we see that the process of finding the present worth of an annuity, or, as it is termed, evaluation, is rendered easy—no matter how long the time—by using logarithms.

CHAPTER VI

SPECIAL FORMS OF ANNUITIES

§ 74. Ordinary or Immediate Annuities

The annuities heretofore spoken of are payable at the end of each period, and are the kind most frequently occurring. To distinguish them from other varieties, they are spoken of as *ordinary* or *immediate* annuities.

§ 75. Annuities Due

When the instalments of an annuity are payable at the beginning of their respective periods, the annuity is called an annuity *due,* although *prepaid* would seem more natural. It is evident that this is merely a question of dating. The instalments compared with those in § 56 are as follows:

	Immediate Annuity 4 Periods	Annuity Due 4 Periods	Immediate Annuity 5 Periods
Amounts of $1.00	$1.00	$1.03	$1.00
	1.03	1.0609	1.03
	1.0609	1.0927	1.0609
	1.0927	1.1255	1.0927
			1.1255
			$5.3091
			— 1.0000
	$4.1836	$4.3091	$4.3091

Hence, to find the amount of an annuity *due,* for any number of periods, say t periods, find the amount of an *immediate* annuity for $t+1$ periods, and subtract one instalment.

§ 76. Present Worth of Annuities Due

In regard to present worths, the instalments compared with those in § 63 would be as follows:

	Immediate Annuity 4 Periods	Annuity Due 4 Periods	Immediate Annuity 3 Periods
Present Worths of $1.00	$.888487	$.915142	$.915142
	.915142	.942596	.942596
	.942596	.970874	.970874
	.970874	1.000000	
			$2.828612
			+ 1.000000
	$3.717099	$3.828612	$3.828612

Therefore, to find the present worth of an annuity *due* for t periods, find the present worth of an *immediate* annuity for $t - 1$ periods, and add one instalment.

§ 77. Present Worth of Deferred Annuities

A *deferred* annuity is one which does not commence to run immediately, but only after a certain number of periods have elapsed. Thus, an annuity of 5 terms, 4 terms deferred, would commence at the beginning of the fifth period, and continue to the end of the ninth period.

If there were nine terms in the annuity, none being deferred, and if the ratio of increase were assumed to be r and

the present worth of the first term were assumed to be unity, the present worth of the annuity for nine terms would be:

$$1+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\frac{1}{r^4}+\frac{1}{r^5}+\frac{1}{r^6}+\frac{1}{r^7}+\frac{1}{r^8} \text{ (§§ 18, 66)}$$

The present worth of the annuity for four terms would be:

$$1+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}$$

The present worth of the annuity for the five deferred terms would, of course, be the difference between the above two sums, or:

$$\frac{1}{r^4}+\frac{1}{r^5}+\frac{1}{r^6}+\frac{1}{r^7}+\frac{1}{r^8}$$

§ 78. Rule for Finding Present Worth of Deferred Annuity

From the foregoing, we derive the rule: To find the present worth of an annuity for m terms, deferred for n terms, subtract the present worth of an annuity for n terms from the present worth of an annuity for $m+n$ terms.

§ 79. Present Worth of Perpetuities

A perpetual annuity, or a *perpetuity,* is one which never terminates. Its amount is infinity, but its present worth can be calculated at any given rate of interest. If each instalment of an annuity is \$1.00 and the rate 5%, the value of the annuity is such a sum as will produce \$1.00 at that rate. This sum is \$20.00, being \$1.00 ÷ 5%. The compound discount is the entire \$1.00, being for an infinite number of terms. Therefore, the rule of § 66 still holds true: divide the compound discount by the single rate of interest, in order to find the present worth of the annuity.

§ 80. Perpetuity in Stock Purchased for Investment

A share of stock may be treated in the same manner as a perpetuity, provided its dividend is assumed to continue at a fixed rate. If the dividend is \$4.00 per share, and if it is desired to purchase at such a basis as to yield 6% on the investment, the price per share should be \$4.00 ÷ 6%, which equals \$66.67. This price is irrespective of the nominal or par value of the stock. Both in perpetuities and in shares of stock, the price $= c \div i$.

§ 81. When Annuity Periods and Interest Periods Differ

In all of these examples of annuities, it has been assumed that the term or interval between payments is the same length of time as the interest period. It frequently happens, however, that the rate of interest is stated to be so much per year, while the payments are half-yearly or quarterly; or there may be yearly payments, while the desired interest rate is to be on a half-yearly basis. We shall defer the treatment of these latter cases until the subject of nominal and effective rates of interest has been discussed.

§ 82. Varying Annuities

There may also be varying annuities, where the instalment changes by some uniform law. These seldom occur in practice. Where the change is simple, as in arithmetical progression, the total annuity may be regarded as the sum of several partial annuities; otherwise the values must be separately calculated for each term. An annuity running for five terms, as follows: 13, 18, 23, 28, 33, may be regarded as the sum of the following:

(1) an annuity of 13 for 5 terms;
(2) an annuity of 5 for 4 terms;

(3) an annuity of 5 for 3 terms;
(4) an annuity of 5 for 2 terms; and
(5) an annuity of 5 for 1 term.

In actual practice, in a case of this kind, in order to find the amount or the present worth of the annuity, it would probably be easiest to find the amount or the present worth of each term, and then find the total of these separate items.

CHAPTER VII

RENT OF ANNUITY AND SINKING FUND

§ 83. Rent of Annuity

The number of dollars in each separate payment of an annuity is called the *rent* of the annuity.

In § 63, we saw that \$3.7171 is the present worth, at 3%, of an annuity composed of 4 payments of \$1.00 each. We may reverse this and say that \$1.00 is the rent of \$3.7171 invested in an annuity of 4 payments at 3%. What, then, is the rent to be obtained by investing \$1.00 in the same way? Since the present worth has been reduced in the ratio of 3.7171 to 1, evidently the rent must be reduced in the same ratio, that is, $1 \div 3.7171$. By ordinary division or by logarithms, this quotient is .26903. Therefore, \$.26903 is the rent of an annuity of 4 terms at 3%, for every \$1.00 invested; or \$1.00 is the present worth at 3% of an annuity for 4 years of \$.26903. This may be illustrated by making up a schedule:

	Rent	Interest	Reduction or Amortization	Value
Beginning of first period.				\$1.00000
End of first period.......	\$ .26903	\$.03	\$ .23903	.76097
End of second period.....	.26903	.02283	.24620	.51477
End of third period......	.26903	.01544	.25359	.26118
End of fourth period.....	.26903	.00785	.26118	0.
	\$1.07612	\$.07612	\$1.00000	

§ 84. Rule for Finding Rent of Annuity

To find the rent of an annuity valued at $1.00, divide $1.00 by the present worth of an annuity of $1.00 for the given rate and time. Rent $= 1 \div P$; and since, by § 67, $P = D \div i$, and $P = A \div a$, we obtain two other symbolic rules:

$$\text{Rent} = i \div D$$
$$\text{Rent} = a \div A$$

§ 85. Alternative Method of Finding Rent

An alternative method of determining the value of the rent of an annuity is to form a proportion, as in arithmetic, and then solve the proportion. For example:

Rent of Annuity				Present Worth of Annuity		
$1.00	:	x	::	$3.7171	:	$1.00

In other words, if a rent of $1.00 produces a present worth of $3.7171, then what quantity of rent will produce a present worth of $1.00? Multiplying the two extremes together, and dividing the product by the mean, we find the other mean to be $.26903, which is the rent required.

§ 86. Rent of Deferred Payments

The problem of finding the rent of an annuity may be regarded as equivalent to another problem—that of finding how much per period for n periods, at the rate i, can be bought for $1.00. A borrower may agree to pay back a loan in instalments, each of which comprises both principal and interest. Suppose that a loan of $1,000 were made under the agreement that such a uniform sum should be paid annually as would pay off (amortize) the entire debt with 3% interest in 4 years. The present worth is, of course, $1,000, and by the above process each instalment or contribution would be $269.03. In countries imposing an in-

come tax, it is usual to incorporate in agreements of this nature a schedule showing what part of the instalment is interest—since that alone is taxable—somewhat as follows:

	Annual Instalment	Interest on Balance	Payment on Principal	Principal Outstanding
January 1, 1914				$1,000.00
December 31, 1914	$269.03	$30.00	$239.03	760.97
December 31, 1915	269.03	22.83	246.20	514.77
December 31, 1916	269.03	15.44	253.59	261.18
December 31, 1917	269.03	7.85	261.18	0.
	$1,076.12	$76.12	$1,000.00	

§ 87. Annuities as Sinking Funds

One other question arises with regard to annuities, and that is in the case of an annuity so constructed as to accumulate to a certain amount at a certain time. The amount to be accumulated is called a *sinking fund.* Frequently the uniform periodical contribution is itself called the sinking fund, but, more strictly speaking, it should be called the sinking fund contribution.

In the case exhibited in the schedule of § 86, the debt was amortized, with the assent of the creditor, by gradual payments. Let us suppose, however, that the creditor prefers to wait until the day of maturity, and receive his $1,000 all at one time, instead of by partial payments. The debtor must pay interest amounting to $30.00 each year, but, in addition to this, in order to provide for the principal on a sinking fund plan, he must transfer from his general assets to a special account (or into the hands of a trustee) such an annual sum as will accumulate, in 4 years at 3%, to $1,000. Since $1.00, set aside annually, amounts, after 4

years on a 3% basis, to \$4.183627 (§ 56), to find what sum will similarly amount to \$1,000, we must divide 1,000 by 4.183627. In this manner the sinking fund contribution is found to be \$239.03.

§ 88. Rule for Finding Sinking Fund Contributions

To find what annuity will amount to \$1.00, or what should be each sinking fund contribution to provide for \$1.00: divide \$1.00 by the amount of an annuity of \$1.00 for the given rate and time. In symbols, sinking fund contribution, or S. F. C., $= 1 \div A$; or (since $A = I \div i$, per § 60) it also equals $i \div I$.

Put in the form of a proportion, the question of § 87 would appear as follows:

Sinking Fund Contribution				Sinking Fund		
\$1.00	:	x	::	\$4.183627	:	\$1,000.00

The unknown quantity, x, would be the same as before, \$239.03.

§ 89. Verification Schedule

The correctness of the result found in § 87 may be proved by a schedule constructed in the following manner:

	Annual Sinking Fund Contribution	Interest During Preceding Year	Total Addition to Sinking Fund	Total Amount Accumulated in Sinking Fund
January 1, 1914				\$ 0.
December 31, 1914	\$239.03		\$239.03	239.03
December 31, 1915	239.03	\$ 7.17	246.20	485.23
December 31, 1916	239.03	14.56	253.59	738.82
December 31, 1917	239.03	22.15	261.18	1,000.00
	\$956.12	\$43.88	\$1,000.00	

§ 90. Amortization and Sinking Fund

On comparing the schedules in §§ 86 and 89, we find that the annual instalments or contributions are respectively \$269.03 and \$239.03, the difference of which is \$30.00, or exactly the yearly interest on the original loan of \$1,000.00. Hence, the amount paid in the second case, if interest be included, is just the same as in the first case. Gradual payments on account of a debt, or gradual accumulations having in view one single final payment in full, therefore amount to the same thing.

As a provision for liquidating indebtedness, or for replacing vanishing assets, sinking fund and amortization are two different applications of the same principle. Formerly, the terms were used interchangeably, but more recently they are distinguished as follows:

(1) The sinking fund method permits the debt to stand until maturity, but in the meantime accumulates a fund which at maturity pays off the entire debt, the interest on the original sum being paid separately.

(2) The amortization method accumulates nothing, but *gradually* reduces the debt, the amount of the reduction being the excess of the periodical payment over the periodical interest.

CHAPTER VIII

NOMINAL AND EFFECTIVE RATES

§ 91. Explanation of Terms

In the previous chapters, all of our computations regarding interest have been based upon a certain number of periods and upon a certain rate per period. In the business world, it is usual to speak of interest rates as so much per annum. In the vast majority of instances, however, the interest, although it is either designated or understood to be per annum, is, nevertheless, not paid by the year (that is, once a year), but in semi-annual or quarterly instalments. Where the interest is payable otherwise than annually, the rate per annum is only nominally correct. For example, if on May 1, 1914, we lend $1,000.00 at 6%, interest to be paid semi-annually, the interest account for the year would be as follows:

November 1, 1914, Interest earned.................	$30.00
May 1, 1915, Interest earned:	
On original loan..........................	30.00
On the $30.00 received on November 1, 1914, for 6 months at an assumed rate of 6%....	.90
Total................................	$60.90

The total interest earnings during the year, therefore, would be $60.90, which is at the *effective* rate of 6.09% on the original investment, as compared with a *nominal* rate of 6%.

§ 92. Semi-Annual and Quarterly Conversions

In the example given in the preceding section, the interest is payable (or, as it is frequently called, convertible) semi-annually. The true or effective rate for each half-yearly period is therefore 3%, and the ratio of increase is 1.03. The amount at the end of the year would be the square of 1.03, or 1.0609, thus giving 6.09% as the effective rate per annum. In the case of quarterly conversion, the amount at the end of the year would be the fourth power of 1.015, or 1.061364, giving 6.1364% as the effective annual rate. The following table shows the effective annual rates for various periods of conversion, the nominal annual rate being 6%:

Period of Conversion	Effective Annual Rate	
Yearly	$1.06 - 1$	or 6.0000%
Semi-annually	$1.03^2 - 1$	or 6.0900%
Quarterly	$1.015^4 - 1$	or 6.1364%
Monthly	$1.005^{12} - 1$	or 6.1678%
Daily..................	$\left(1 + \frac{.06}{365}\right)^{365} - 1$	or 6.1826%

§ 93. Limit of Effective Annual Rate

It will be seen that the effective rate increases as the conversions become more frequent. There is a limit, however, beyond which this acceleration will not go. If an investment on a 6% nominal annual rate is compounded every minute, or every second, or every millionth of a second, or constantly, the effective annual rate could never be so great as 6.184%.*

*See § 238.

§ 94. Rule for Effective Rate

From observation of the table shown in § 92, we may deduce the following symbolic rule for finding the effective rate, m representing the number of payments per annum, and j the effective rate:

$$j = \left(1 + \frac{i}{m}\right)^m - 1$$

§ 95. Logarithmic Process

In order to exemplify logarithmic processes in working out the foregoing rule, let it be required to find the effective rate of interest when the nominal rate is 6% per annum, compounded daily. The rule in § 94 then becomes:

$$j = \left(1 + \frac{.06}{365}\right)^{365} - 1$$

By the use of logarithms, we obtain:

$$\begin{aligned} \log.\ .06 &= \bar{2}.7781513 \\ \log.\ 365 &= 2.5622929 \\ \text{Hence, } \log.\ (.06 \div 365) &= \bar{4}.2158584 \end{aligned}$$

$\bar{4}.2158584$ is, we find, the logarithm of .0001643835; and, therefore, the value found thus far is:

$$j = (1 + .0001643835)^{365} - 1$$
$$\text{or, } j = 1.0001643835^{365} - 1$$

The logarithm of 1.0001643835 is .00007138; and 365 times this latter figure is .02605370, which we find to be the logarithm of 1.061826. The value for the effective rate then becomes:

$$j = 1.061826 - 1$$
$$\text{or, } j = .061826, \text{ or } 6.1826\%$$

CHAPTER IX

BONDS AND THE PROPER BASIS OF BOND ACCOUNTS

§ 96. Provisions of Bonds

The most common forms of interest-bearing securities are bonds. Every bond contains a complex promise to pay:

(1) A certain sum of money at a stipulated future time, this sum being known as the principal, or par.
(2) Certain smaller sums, proportionate to the principal, and payable at various earlier times than the principal.

These smaller sums are usually known as the interest payments, but, as they do not necessarily correspond to the true rate of interest, it will be better to speak of them as the coupons.

Bonds also contain provisions as to the time, place, and manner of these payments, and usually refer, also, to the mortgage, if any, made to insure their fulfillment, and to the law, if any, authorizing the issue.

§ 97. Interest on Bonds

The rate of interest named in a bond is usually an integer per cent, or midway between two integers: as, 2%, 2½%, 3%, 3½%, 4%, 4½%, 5%, 6%, 7%, etc. Occasionally such odd rates occur as 3¼%, 3.60%, 3.65%, 3¾%, but these are unusual and inconvenient. Most bonds provide for semi-annual payments of interest; a considerable number of

Principle = PV of prospective benefits –

issues, however, pay interest quarterly, and a very few annually. With most bonds, the interest is payable on the first day of the month. In the case of a very few bonds the interest falls due on the 15th or on the last day of the month. In some respects it would be better if bond interest were payable on the last day of a calendar month, instead of on the first day of the succeeding month, since the entire transaction (including the payment of cash for the accrued interest) would thus be brought inside of a calendar period. The item of "Interest Accrued" on monthly balance sheets would in this manner frequently be eliminated, or at least substantially reduced.

§ 98. How Bonds Are Designated

Bonds are usually designated according to the obligor, the rate of interest, the date of maturity, and sometimes the initials of the months when interest is payable. Thus, "Manhattan 4's of 1990, J J" indicates the bonds of the Manhattan Railway Company, bearing 4% interest per annum, the principal being due in 1990, and the interest coupons being payable semi-annually in January and July.

§ 99. Relation of Cost to Net Income

Bonds are seldom bought or sold at their exact par value, and this fact has an effect on the rate of net income derived from the original investment. If the amount invested is greater than the par value, the difference is known as the premium. This premium is not repaid at maturity, as is the par value or principal of the bond, and hence must be provided for out of the various interest payments. Thus, a bond purchased above par produces a lower rate of income than the rate of interest represented by the coupons. Conversely, if the purchase is below par, the investor will, at maturity, receive not only the amount of his original in-

vestment, but also the difference between this amount and the par value of the bond. This difference, technically known as the discount, has the effect of making the rate of income higher than the rate of interest shown by the coupons.

§ 100. Coupon and Effective Rate of Interest on Bonds

The following are some of the expressions used to denote an investment made above par: "6% bond to net 5%"; "6% bond on 5% basis"; "6% bond yielding 5%"; "6% bond paying 5%"; etc. In the cases of bonds bought below par, the income rate would be larger than the coupon rate, as, for example, "3% bond to net 4%," etc. In all of the above instances, the percentage immediately preceding the word "bond" signifies the coupon rate of interest, while the other percentage signifies the true or effective rate of interest.

§ 101. Present Worth of Bonds

It will be seen, therefore, that the sale of a bond involves the transfer of the right to receive, at the stipulated times, both the principal and the periodical amounts of interest. None of these various sums is ever worth its face value, or par, until the arrival of its stipulated date of payment. The principal is never worth its face value until its maturity, and the coupons are never worth their face values until their respective maturities. Yet, while both principal and coupons are always at a discount, except at their respective dates of maturity, the aggregate value or present worth of the principal and coupons at any one time prior to maturity is frequently more than the par value of the principal alone (as in the case of a bond bought at a premium); and it is this aggregate present worth of both principal and coupons which is always the question at issue in connection with the purchases and sales of bonds.

§ 102. Considerations in the Purchase of Bonds

In fixing the price which he is willing to pay, the purchaser is guided by several considerations, among them the following:

(1) The amount of the principal.
(2) The date of maturity of the principal.
(3) The amount of each coupon.
(4) The number of coupons.
(5) The dates of maturity of the various coupons.
(6) The rate of interest which can be earned upon securities of a similar grade.

This last point also involves a determination of the degree of probability that the principal and the various coupons will be promptly paid at their dates of maturity; or, in other words, consideration must be given to the financial reputation and integrity of the obligor.

§ 103. Present Worth and Earning Capacity of Bonds

In effect, the purchaser of a bond discounts, at a certain fixed rate, the principal and each coupon at compound interest, for the periods which they respectively have to run, and the sum of these partial present worths is the value of the bond. If he can buy at a price below this value, he will receive a higher rate of interest than he anticipated. If he has to pay more than this value, his rate of interest will be lower. As he cashes each coupon, he receives what he paid for it, plus compound interest at the uniform rate; thenceforward he earns interest on a diminished investment as far as coupons are concerned, but on an increased investment as to principal. If the par value of his coupons is less than the total interest earned during the period, there is an increase in the total investment; if such par value is greater, then there is a surplus which operates to reduce the investment or to amortize the premium.

§ 104. Cost and Par of Bonds

There are, therefore, two fixed points in the history of a bond: the original cost, or money invested, and the principal, or par—the money to be received at maturity. Between these two points there is a gradual change: if bought below par, the bond must rise to par; if bought above par, it must sink to par. This gradual change is the resultant effect of two opposing forces, the interest earned tending to increase the investment value, while the payment of coupons reduces the investment value. At any intermediate moment between these two points there is an investment value which can be calculated, and which is just as true as the original cost and the par. In fact, these latter are merely special cases of investment value; the investment value at the date of purchase is cost, and at the date of maturity it is par.

§ 105. Intermediate Value of Bonds

The gradual change in investment value of bonds between purchase and maturity is ignored by some investors, who, during the whole period, use either the original cost or the par value. In the former case they suppose that the investment value remains at its original figure until the very day of maturity, and is then instantly changed to par, either by a loss of all of the premium or by a sudden gain of all of the discount. Those who use par as the investment value also assume that there is this sudden change of value, the difference being that the change occurred at the instant of purchase instead of at maturity. These methods of treatment are manifestly fictitious and unreal, and are only resorted to on account of the labor involved in computing intermediate values. Experience would tell us, if theory did not, that there is no such violent change. The cost and the par value, while entirely correct at the beginning and at the end, respectively, of the period of ownership, are entirely incorrect during the interim.

§ 106. True Investment Basis for Bonds

The true standard of investment value for bonds is the present worth, at compound interest, of all recipiends, or sums of cash to be received, whether such sums be called coupons or principal. Neither the original cost of a bond nor its ultimate par is a proper permanent investment basis. The bond should enter into the accounts at cost, which is a fact, and should go out of the accounts at par, which is another fact. During the interim, the change from cost to par should take place gradually by the processes of amortization or accumulation, at the rate of the true interest on the original investment.

§ 107. Various Bond Values

There are thus three values in the life of a bond which resemble three tenses in grammar: The *past* tense represents the cost, that is, the amount originally paid; the *future* tense represents the par, which is the amount ultimately to be received; while the *present* tense represents the investment value, intermediate between the values of the past and future, except in the special case of a bond bought at par.

There is also a fourth value of a bond, that is, the amount which might be obtained on sale at the present time. This is the market value, and is a matter of judgment, opinion, and inference. Although the market value of a bond has great utility in some respects, it has no place, strictly speaking, in accounts kept with regard to investments. It is not an act or a fact of the business; it is a statement of what *might* be done. The market value contemplates a possibility, or a probability—but never an actuality, in so far as the accounts are concerned, unless a sale is actually consummated. If an investor has had an opportunity to make a sale of a bond, but has allowed it to pass by, the mere fact that he has been offered such an op-

portunity to sell has not the slightest effect on his financial status.

§ 108. Investment Value the True Accounting Basis

Unless accounts with respect to bonds and similar securities are kept on the investment-value basis, an investor is unable to tell whether a contemplated selling price will result in a loss or a gain. If the books are kept on the basis of par, every sale above par will appear as a gain, even though it may be a losing bargain; while a comparison with the original cost will be equally delusive and unsatisfactory.

CHAPTER X

VALUATION OF BONDS

§ 109. Cash Rate and Income Rate of Bonds

With respect to all bonds bought above or below par, there are always two rates of interest involved: first, a *nominal* or *cash* rate, which is a certain percentage of par, and which is indicated by the coupons; and second, an *effective* or *income* rate, which is a certain percentage of the amount originally invested and remaining invested. For the sake of greater clearness, we shall use the terms cash rate and income rate, since they are more readily understood than the terms nominal and effective. The symbols c and i will respectively designate the cash rate and the income rate. $1+i$ is the ratio of increase as heretofore. The symbol $1+c$ will not be required, since c is not an accumulative rate, but merely an annuity purchased with the bond, the number of periods of the annuity being the same as the number of coupons attached to the bond. The difference of rates is $c-i$, or $i-c$.

§ 110. Elements of a Bond

In a bond purchased above or below par, we have, therefore, the following elements: the par, or principal, payable after n periods; an annuity of c per cent of par for n periods; and a ratio of increase, $1+i$. With these elements given, there are two distinct methods for finding the value of the entire security, and these must give the same result.

§ 111. Valuation of Bonds—First Method

As an illustration of this method, let us take the case of a 7% bond, having 25 years (50 periods) to run, interest payable semi-annually, the par being $1,000. Suppose that it is required to compute the value of the bond at the beginning of its first interest period. This present value is composed of two parts: (a) the present worth of $1,000 due 50 periods hence; and (b) the present worth of an annuity of $35 for 50 terms. We cannot ascertain the value of these two parts until we know the income rate current upon securities of a similar grade. Let us assume that this income rate is 3% per period, or what is usually called a 6% basis. The ratio of increase is thus 1.03 per period.

§ 112. (a) Finding Present Worth of Principal

The first part of the solution is to find the present worth of $1,000 due in 50 periods, at 3% per period. In § 52, we have found the present worth of $1.00, under the same conditions, to be $.2281071; hence the similar present worth of $1,000 is $228.1071. This result, it will be noticed, has not the slightest reference to the 7% rate of the bond. For the purposes of the first part of the solution, the cash or coupon rate is absolutely immaterial; the bond might be equally well a 10% bond or a 0% bond, in the latter case bearing no coupons at all.

§ 113. (b) Present Worth of Coupons

We next have to find the present value of an annuity of $35 for 50 terms at 3%. In § 73, we found the present value of a similar annuity of $1.00 to be $25.72976 +. An annuity of $35, therefore, has a present value of $900.5417. Hence, we have the following:

Present worth of the par............	$228.1071
Present worth of the coupons........	900.5417
Present worth of the entire bond....	$1,128.6488

The ordinary tables, which give the values of a $100 bond only, read $112.86, which is the same as the above, rounded off. The above computation gives a result which is correct to the nearest cent on $100,000, viz.: $112,864.88.

§ 114. Schedule of Evaluation

In order to present the subject still more clearly, in a schedule form, let it be required to find the value, as at January 1, 1913, of a 7% bond for $1,000, interest payable semi-annually, due at January 1, 1915, the income rate being 3% per period. In § 30, we have found that the present worth of $1.00 for 1, 2, 3, and 4 periods is $.970874, $.942596, $.915142, and $.888487, respectively. The respective present worths of $35.00 are, therefore, $33.980590, $32.990860, $32.029970, and $31.097045. The following schedule may then be formed:

Items to be Evaluated	Dates of Maturity	Periods from Jan. 1, 1913, to Dates of Maturity	Present Worth at Jan. 1, 1913
Coupon, $35	July 1, 1913	1	$33.980590
" 35	January 1, 1914	2	32.990860
" 35	July 1, 1914	3	32.029970
" 35	January 1, 1915	4	31.097045
Total			$130.098465
Par, $1,000	January 1, 1915	4	888.487
Grand Total..........			$1,018.585465

The total present value of the four coupons ($130.098465) could have been found by one operation, as was done in the preceding section, and this is the usual method of finding the present worth of an annuity. The foregoing schedule, however, sets forth the details clearly, although it is not a practicable method of evaluation when the number of coupons is large.

§ 115. Valuation of Bonds—Second Method

In illustration of this method, we shall assume the same facts as presented in § 111. Each semi-annual payment of $35 may be considered as made up of two parts: $30 and $5. The $30 is the income on the $1,000 par value at the assumed semi-annual income rate of 3%. We may disregard this, and consider only the $5, which is a surplus over and above the income rate, and, in fact, is an annuity which must be paid for and which is represented by the premium paid on the bond. Having devoted $30 to the payment of our expected income-rate on par, we have a remainder of $5, the difference in rates per period; this annuity of $5, in excess of the income rate, is a semi-annual benefit the value of which is to be ascertained.

We have already found the present value of an annuity of $1.00 for 50 terms at 3% to be $25.72976. The present value of a similar annuity of $5.00 would therefore be $128.6488, which is the premium and which agrees with the result found in § 113. The method is not only quicker than the first method presented, but also often gives one more place of decimals.

§ 116. Evaluation when Cash Rate Is Less than Income Rate

In the case of a bond sold below par, and where, accordingly, the cash rate is less than the income rate, the

same procedure is followed for finding the present worth of an annuity of the difference in rates. In the above illustration, if the bond had a cash rate of 5% instead of 7%, the annuity to be evaluated would still be $5 (that is, $30 less $25). In this case, however, the value of the annuity ($128.6488) would have to be subtracted from the par, giving $871.3512 as the value of a 5% bond, due in 25 years and having an income rate of 3% per period. This would be commonly known as a 6% basis, although the effective annual income is 6.09%, as pointed out in § 91.

§ 117. Second Method by Schedule

As a further illustration of the second method of evaluation, let us take the case of the bond described in § 114. Under the second method the schedule would be:

Differences Between Cash and Income Rates ($c-i$)	Dates of Maturity	Periods from Jan. 1, 1913, to Dates of Maturity	Present Worth at Jan. 1, 1913
$5	July 1, 1913	1	$4.854370
5	January 1, 1914	2	4.712980
5	July 1, 1914	3	4.575710
5	January 1, 1915	4	4.442435
Total			$18.585495

The premium above found disagrees slightly with that shown in § 114, since in the latter case there is a loss of three decimal places in finding the present worth of the $1,000 par value. In examining the above schedule, it must be borne in mind that the total can be ascertained by a single operation, and that the details are here presented only for the sake of additional clearness.

§ 118. Rule for Second Method of Evaluation

Since the second method is superior to the first, it will hereafter be considered as the standard; and we give accordingly the following rule: The premium (or discount) on a bond bought above (or below) par is the present worth, at the income rate, of an annuity equal to the difference between the cash and income rates for the life of the bond.

§ 119. Principles of Investment

We have found the value of a 7% bond for $1,000, paying 6% (semi-annually), due in 25 years, to be $1,128.65 to the nearest cent. This is the amount which must be invested if the 6% income is to be secured. At the end of the first half-year, the holder of the bond receives, as income, 3% interest on the $1,128.65 originally invested, which is $33.86. But he actually collects $35.00, and after deducting $33.86 as revenue, there remains $1.14, which must be applied in amortizing the premium. This will leave the value of the bond at the end of the first half-year, at the same income rate, $1,127.51. If our operations have been correct, the value of a 7% bond to net 6% (payable semi-annually), having 24½ years or 49 periods to run, will be $1,127.51. To test this, and to exemplify the method through the use of logarithms, the entire operation is presented in the following section.

§ 120. Solution by Logarithms

The logarithm of 1 is....................	zero
The logarithm of 1.03 is..................	.01283722
The logarithm of 1.03^{49} is therefore........	.6290238
The logarithm of $(1 \div 1.03^{49})$ is therefore..	$\bar{1}.3709762$
We find that the logarithm of .23495 is......	$\bar{1}.3709754$
Remainder	8

This gives the additional decimal figures 02.

Hence, \$.2349502 is the present value of \$1.00 at 3% per period for 49 periods. The compound discount is therefore \$.7650498, and this divided by the single rate of interest, 3%, gives the result \$25.50166, which is the present value of an annuity of \$1.00 per period. The difference between the cash and income rates is \$5, i.e., \$35 — \$30. Therefore, the present value of a \$5 annuity for 49 periods at 3% would be \$127.508, or, rounded off, \$127.51, which is the premium desired. Adding this to the par, we have \$1,127.51, which agrees with the result obtained in § 119.

§ 121. Amortization Schedule

When bonds are purchased for investment purposes, a *Schedule of Amortization* should be constructed, showing the gradual extinction of the premium by the application of the surplus interest. The form shown below is recommended for this purpose, although it is merely suggestive and not complete. The calculations should be continued to the date of maturity, and at intervals corrected in the last figure by a fresh logarithmic computation.

Schedule of Amortization

7% Bond of the......................, payable January 1, 1939. Net 6%. J J.

Date	Total Interest 7%	Net Income 6%	Amortization	Book Value	Par
1914, Jan. 1	Cost......	....	....	..\$1,128.65	\$1,000.00
July 1	\$35.00	\$33.86	\$1.14	1,127.51	
1915, Jan. 1	35.00	33.83	1.17	1,126.34	
July 1	35.00	33.79	1.21	1,125.13	
		etc., etc., etc.			

Strictly speaking, the net income rate is not 6% per annum, but 3% for each semi-annual period, or an effective

annual rate of 6.09%. The column headed "Total Interest" could be changed to "Cash Receipts," and the term "Book Value" might also be called "Investment Value."

§ 122. Use of Schedules in Accountancy

The foregoing schedule is the source of the entry which should be made each half-year for "writing off" the premium or "writing up" the discount, in order that at maturity the bond may stand exactly at par. Two other schedules are set forth below, in which the semi-annual steps in the changing value of the bond are shown in detail from the date of purchase until maturity, one schedule being for a bond bought above par, and the other for a bond bought below par. Since the formation of schedules is the basis of the accountancy of amortized securities, we shall present the same material in various forms, and shall attach to the schedules the letters (A), (B), etc., for the purposes of ready reference.

SCHEDULE (A)—AMORTIZATION

5% Bond of the...................., payable May 1, 1919. M N.

Date	Total Interest 5%	Net Income 4%	Amortization	Book Value	Par
1914, May 1	Cost......	....	....	$104,491.29	$100,000.00
Nov. 1	$ 2,500.00	$ 2,089.83	$ 410.17	104,081.12	
1915, May 1	2,500.00	2,081.62	418.38	103,662.74	
Nov. 1	2,500.00	2,073.26	426.74	103,236.00	
1916, May 1	2,500.00	2,064.72	435.28	102,800.72	
Nov. 1	2,500.00	2,056.01	443.99	102,356.73	
1917, May 1	2,500.00	2,047.13	452.87	101,903.86	
Nov. 1	2,500.00	2,038.08	461.92	101,441.94	
1918, May 1	2,500.00	2,028.84	471.16	100,970.78	
Nov. 1	2,500.00	2,019.42	480.58	100,490.20	
1919, May 1	2,500.00	2,009.80	490.20	100,000.00	
	$25,000.00	$20,508.71	$4,491.29		

SCHEDULE (B)—ACCUMULATION

3% Bond of the.................., payable May 1, 1919. M N.

Date	Total Interest 3%	Net Income 4%	Accumulation	Book Value	Par
1914, May 1				$95,508.71	$100,000.00
Nov. 1	$ 1,500.00	$ 1,910.17	$ 410.17	95,918.88	
1915, May 1	1,500.00	1,918.38	418.38	96,337.26	
Nov. 1	1,500.00	1,926.74	426.74	96,764.00	
1916, May 1	1,500.00	1,935.28	435.28	97,199.28	
Nov. 1	1,500.00	1,943.99	443.99	97,643.27	
1917, May 1	1,500.00	1,952.87	452.87	98,096.14	
Nov. 1	1,500.00	1,961.92	461.92	98,558.06	
1918, May 1	1,500.00	1,971.16	471.16	99,029.22	
Nov. 1	1,500.00	1,980.58	480.58	99,509.80	
1919, May 1	1,500.00	1,990.20	490.20	100,000.00	
	$15,000.00	$19,491.29	$4,491.29		

§ 123. Book Values in Schedules

In the foregoing two schedules, (A) and (B), it will be observed that at any given date the book value in Schedule (A) is always exactly as much above par as the book value in Schedule (B) is below par. During any given period, the "amortization" and the "accumulation" are exactly the same in both, being deducted in Schedule (A) and added in Schedule (B).

§ 124. Checks on Accuracy of Schedules

There are three internal checks which are of value in verifying the accuracy of the schedules. For example, in Schedule (B), the following facts may be observed:

(1) The total interest plus the total accumulation equals the total net income.

(2) The total accumulation equals the par less the book value; or, in other words, it equals the inaugural discount.

(3) Each item of accumulation equals the preceding one multiplied by the semi-annual ratio of increase 1.02, the semi-annual net income being 2%. That is:

$$\$461.92 \times 1.02 = \$471.16$$
$$\$471.16 \times 1.02 = \$480.58$$
etc.

In some instances in these computations, there will be an apparent error of one cent, which is accounted for by the fact that the number of decimal places is not carried out sufficiently far.

§ 125. Tables Derivable from Bond Values

The figures in the column headed "Book Value" might be taken from tables of bond values published in book form. If Sprague's Eight-Place Bond Tables were used, and if the column "Book Value" were copied directly from the tables, the other columns could be derived by the processes of addition or subtraction. The result arrived at by this method would be exactly the same as the results shown in Schedules (A) and (B). The successive amounts of amortization or accumulation would be found by finding the differences between successive book values; while the net income for any period would be found by either adding the accumulation to the total interest, or by deducting the amortization from the total interest.

§ 126. Methods of Handling Interest

It will be observed that in Schedules (A) and (B), the entire interest is accounted for, both in the case of the interest on par plus premium, and also in the case of the interest on par minus discount. We may easily construct the schedules so as to eliminate the par and the interest thereon

at the rate i. In this manner we would have to deal only with the surplus interest or the deficient interest, according to the theory explained in § 115. Since this method may be preferable for some forms of accounts, a new schedule is presented below, based on the same facts as those shown in Schedule (A):

SCHEDULE (C)—AMORTIZATION; PREMIUM ONLY

Date	Surplus Interest on Par 1%	Interest on Premium 4%	Amortization	Premium
1914, May 1				$4,491.29
Nov. 1	$ 500.00	$ 89.83	$ 410.17	4,081.12
1915, May 1	500.00	81.62	418.38	3,662.74
Nov. 1	500.00	73.26	426.74	3,236.00
1916, May 1	500.00	64.72	435.28	2,800.72
Nov. 1	500.00	56.01	443.99	2,356.73
1917, May 1	500.00	47.13	452.87	1,903.86
Nov. 1	500.00	38.08	461.92	1,441.94
1918, May 1	500.00	28.84	471.16	970.78
Nov. 1	500.00	19.42	480.58	490.20
1919, May 1	500.00	9.80	490.20	0.
	$5,000.00	$508.71	$4,491.29	

§ 127. Schedule of Bond Values

Another way of setting forth the value of bonds at the successive interest dates is shown in the following table, which indicates clearly the steps taken in computing the value. This table, however, is not nearly so compact as the preceding ones, and for this reason is not recommended, for most purposes. We will take as an illustration Schedule (A), shown in § 122.

Value of bond at May 1, 1914 (cost)		$104,491.29
Amortization for ensuing 6 months:		
Nominal interest at $2\frac{1}{2}\%$ on $100,000.00	$2,500.00	
Effective interest at 2% on $104,491.29	2,089.83	
Difference, being the amortization to be subtracted from the investment value		410.17
Value of bond at November 1, 1914		$104,081.12
Amortization for ensuing 6 months:		
Nominal interest at $2\frac{1}{2}\%$ on $100,000.00	$2,500.00	
Effective interest at 2% on $104,081.12	2,081.62	
Difference, being the amortization to be subtracted from the investment value		418.38
Value of bond at May 1,1915		$103,662.74

etc., etc.

A slight variation of the above form is to put all of the figures of the schedule in one column, as follows:

Value, May 1, 1914	$104,491.29
Plus effective interest	2,089.83
	$106,581.12
Minus amortization	2,500.00
Value, November 1, 1914	$104,081.12
Plus effective interest	2,081.62
	$106,162.74
Minus amortization	2,500.00
Value, May 1, 1915	$103,662.74

etc., etc.

By using red ink for the subtrahends (which are indicated by italic figures), the addition and subtraction can be performed at one operation, viz.:

$104,491.29
2,089.83
2,500.00
$104,081.12
2,081.62
2,500.00
$103,662.74
2,073.26
2,500.00
$103,236.00
etc., etc.

It will be noticed that the computation of the interest may be done without using any other paper. Even with a fractional rate, such as 2.7% per annum, or 1.35% per period, the 1%, the .3%, and the .05% may be successively written down direct without further computation. For example:

Assumed inaugural value..........	$120,039.00
	1,200.39
	360.117
	60.019
	2,500.00
Value at end of 6 months..........	$119,159.526
	1,191.595
	357.479
	59.580
	2,500.00
Value at end of 1 year............	$118,268.180
	etc., etc.

CHAPTER XI

VALUATION OF BONDS (*Concluded*)

§ 128. Bond Purchases at Intermediate Dates

It has hitherto been assumed that the purchase of the bond took place exactly upon an interest date. In the vast majority of purchases, however, the purchase date differs from the interest date, and we will now consider cases of this character. Let us suppose that the interest dates are May 1 and November 1, whereas the purchase took place on July 1, after one-third of the interest period had elapsed. The business custom is to adjust the matter as follows: The buyer pays to the seller the (simple) interest accrued for the two months, acquiring thereby the full interest rights, which will fall due on November 1, and the premium (or the discount, as the case may be) is also considered as vanishing by an equal portion each month, so that one-third of the half-yearly amortization takes place by July 1. Taking as an illustration the bond considered in Schedule (A) (§ 122), the amortization from May 1, 1914, to November 1, 1914, is \$410.17; the amortization up to July 1 would therefore be one-third of this amount, or \$136.72. The book value at July 1 is \$104,491.29 minus \$136.72, plus \$833.33 (the accrued interest for two months), giving a net figure of \$105,187.90. This last amount is called the *flat price,* that is, it is the price including interest; if the interest is not included, the price is said to be at so many per cent *and interest.* These are the two methods in most common use for indicating the prices of bonds. The flat price as above

computed might also have been obtained in the following manner:

To the value on May 1, 1914	$104,491.29
add simple interest thereon for 2 months at 4%, which is the effective income rate	696.61
giving the flat price at July 1, 1914	$105,187.90

§ 129. Errors in Adjusting Bond Prices

This practice of adjusting the price of bonds at intermediate dates by simple interest is conventionally correct, but is scientifically inaccurate, and always works a slight injustice to the buyer. The seller is having his interest compounded at the end of two months instead of six months, and receives a benefit therefrom at the expense of the buyer. It will be readily seen that the buyer does not net the effective rate of 4% semi-annually on his investment of $105,187.90. In order to give both buyer and seller a return at the effective rate of 2% semi-annually (or 4.04% annually), with a bimonthly conversion for the seller and a four-monthly conversion for the buyer, the true price would be $105,183.31.* In practice, however, for any time under six months, simple interest is generally used, to the slight disadvantage of the buyer, who may claim that the value at November 1, ($104,081.12) + interest due ($2,500.00), should have been discounted at 4%. This would give $106,581.12 ÷ 1.01⅓, or $105,178.74. This latter figure is almost exactly as much too low ($4.57) as the $105,187.90 is too high ($4.59).

*This price is found by finding the cube root of 1.02, which is 1.00662271. This last figure is the rate for a two-months period at the effective rate of 2% semi-annually, or 4.04% annually. When the value at May 1 ($104,491.29) is multiplied by this figure, the result is $105,183.31, which is the true price on an effective income basis of 4.04% annually.

§ 130. Schedule of Periodic Evaluation

The schedule would therefore, in practice, read as follows:

SCHEDULE (D)—PERIODIC VALUATION; SIMPLE INTEREST

Date	Total Interest 5%	Net Income 4%	Amortization	Book Value	Par
1914, July 1	Cost..............			$104,354.57	$100,000.00
Nov. 1	$ 1,666.67	$ 1,393.22	$ 273.45	104,081.12	
1915, May 1	2,500.00	2,081.62	418.38	103,662.74	
Nov. 1	2,500.00	2,073.26	426.74	103,236.00	
1916, May 1	2,500.00	2,064.72	435.28	102,800.72	
Nov. 1	2,500.00	2,056.01	443.99	102,356.73	
1917, May 1	2,500.00	2,047.13	452.87	101,903.86	
Nov. 1	2,500.00	2,038.08	461.92	101,441.94	
1918, May 1	2,500.00	2,028.84	471.16	100,970.78	
Nov. 1	2,500.00	2,019.42	480.58	100,490.20	
1919, May 1	2,500.00	2,009.80	490.20	100,000.00	
	$24,166.67	$19,812.10	$4,354.57		

§ 131. Objection to Valuation on Interest Dates

The interest dates may not always be the most convenient dates for periodical valuation. In the case of an investment consisting of several kinds of bonds, there would generally be some interest coupons falling due in every month of the year, and yet on a certain annual or semiannual date the entire holdings must be simultaneously valued, irrespective of the varying interest dates. In cases of this kind, it will therefore be convenient if the schedules can be arranged in such a manner that, without recalculation, every book value will be ready to place in the balance sheet. Fortunately, this is easier than would be supposed.

§ 132. Interpolation Method of Periodic Valuation

As an illustration, we will again take the bond described in § 130; but we will now assume that the investor closes his books on the last days of June and December. We will suppose that the purchase is made on August 1, 1914. Since August 1 is midway between May 1 and November 1, the price must be adjusted as shown in § 128. The price at August 1 would therefore be midway between \$104,491.29 and \$104,081.12—namely, \$104,286.20—plus, of course, the accrued interest (\$1,250.00), this being the customary, not the theoretical, method. The value at November 1 need not enter into the schedule, but we must compute the December 31 value in the same manner as we found the July 1 value in § 128. One-third of the difference between \$104,081.12 and \$103,662.74, or \$418.38, is \$139.46; \$104,081.12 — \$139.46 = \$103,941.66. Our schedule so far, the headings being the same as in § 130, reads:

1914, Aug. 1	Cost........................				\$104,286.20	\$100,000.00
Dec. 31		\$2,083.33	\$1,738.79	\$344.54	103.941.66	

Proceeding in the same way to find the value on June 30, 1915, from those on May 1 and November 1, we get \$103,520.49. To these values at dates when interest does not fall due, there must be added the accrued interest to find the total values. This method of finding the value of bonds between interest dates is called interpolation.

§ 133. Multiplication Method of Valuation

There is another method of finding the intermediate values, however, which might be called the multiplication method. Having found the value at December 31 to be \$103,941.66, the interest for six months thereon at 4% is \$2,078.83, which, subtracted from the coupon interest

(\$2,500.00), gives as the amortization \$421.17. This latter amount, written off from \$103,941.66, gives \$103,520.49 as the value at June 30, which is precisely the same result as was obtained by interpolation between May 1 and November 1 in § 132. In practice, the method of multiplication will be found more convenient than the method of interpolation. Having once adjusted the value at one of the balancing periods, we can derive all of the values at the remaining balancing periods by finding the net income, subtracting it from the cash interest and reducing the premium by the difference, completely ignoring the values on interest days (M N).

§ 134. Computation of Net Income for Partial Period

No difficulty arises until we reach the broken period, January 1 to May 1, 1919. Here the computation of the net income is peculiar; the par and the premium must be treated separately. The net income on \$100,000.00 is taken at ⅔ of 2% for the ⅔ time, giving \$1,333.33. The premium, \$326.80, however, must always be multiplied by the full 2%, giving \$6.54. Adding \$1,333.33 and \$6.54, we have \$1,339.87, which, used as heretofore, reduces the principal to par. The reason for this peculiarity is that \$490.20 (the premium at November 1, 1918), and not \$326.80 (the premium at December 31, 1918), is the conventional premium on which 4% is to be computed. Hence, instead of taking \$490.20 for ⅔ of a period, we take \$326.80 itself for a whole period; these two methods reach the same result, since \$490.20 is 3/2 of \$326.80, and two-thirds of three-halves is unity. In other words, ⅔ of a dollar for a whole period is equivalent in value to the whole of the dollar for ⅔ of a period. On the basis outlined, the completed schedule would therefore be as follows:

SCHEDULE (E)—PERIODIC VALUATION BY MULTIPLICATION

Date	Total Interest 5%	Net Income 4%	Amortization	Book Value	Par
1914, Aug. 1	Cost......	....	....	$104,286.20	$100,000.00
Dec. 31	$ 2,083.33	$ 1,738.79	$ 344.54	103,941.66	
1915, Jun. 30	2,500.00	2,078.83	421.17	103,520.49	
Dec. 31	2,500.00	2,070.41	429.59	103,090.90	
1916, Jun. 30	2,500.00	2,061.82	438.18	102,652.72	
Dec. 31	2,500.00	2,053.05	446.95	102,205.77	
1917, Jun. 30	2,500.00	2,044.12	455.88	101,749.89	
Dec. 31	2,500.00	2,035.00	465.00	101,284.89	
1918, Jun. 30	2,500.00	2,025.70	474.30	100,810.59	
Dec. 31	2,500.00	2,016.21	483.79	100,326.80	
1919, May 1	1,666.67	1,339.87	326.80	100,000.00	
	$23,750.00	$19,463.80	$4,286.20		

§ 135. Purchase Agreements

In all the foregoing examples it has been assumed that the bond has been bought "on a basis," which means that the buyer and seller have agreed upon the income rate which the bonds shall pay, and that from this the price has been adjusted. But in probably the majority of cases the bargain is made "at a price," and then the income rate must be found. This is a more difficult problem.

§ 136. Approximation Method of Finding Income Rate*

The best method of ascertaining the basis, when the price is given, is by trial and approximation—in fact, all methods more or less depend upon that. The ordinary tables will locate several figures of the rate, and one more figure can safely be added by simple proportion. But it is an important question to what degree of fineness we should try to attain. It seems to be the consensus of opinion and

*For a new method of approximation, see Chapter XXIII.

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practice that to carry the decimals to hundredths of one per cent is far enough, although in some cases, by introducing eighths and sixteenths, two-hundredths and four-hundredths may be required. Sprague's Tables, by the use of auxiliary figures, give values for each one-hundredth of one per cent.

§ 137. Application of Method

Let us suppose that $100,000 of 5% bonds, 5 years to run, M N, are offered at the round price of 104½ on May 1, 1914. It is evident that this is nearly, but not quite, a 4% basis. Trying a 3.99% basis we find that the premium is $4,537.39, which is further from the price than is $4,-491.29, the 4% basis. Hence, 4% is the nearest basis within 1/100 of one per cent. In fact, by repeated trials, we find that the rate is about .0399812 per annum. It is manifest that such a ratio of increase as 1.0199906 would be very unwieldy and impracticable, and that such laborious exactness would be intolerable. Yet here we have paid $104,500, and the nearest admissible basis gives $104,491.29; what shall be done with the odd $8.71? It must disappear before maturity, and on a 4% basis it will be even larger at maturity than now. Three ways of ridding ourselves of it may be suggested.

§ 138. First Method of Eliminating Residues

Add the residue $8.71 to the first amortization, thereby reducing the value to an exact 4% basis at once. In Schedule (A)—shown in § 122—the first amortization would be $418.88, instead of $410.17. This is at the income rate of about 3.983% for the first half-year and thereafter at 4%. For short bonds the result is fairly satisfactory.

§ 139. Second Method of Eliminating Residues

Divide $8.71 into as many parts as there are periods.

This would give $.87 for each period, except the first, which would be $.88 on account of the odd cent. Set down the 4% amortization in one column, the $.87 in the next, and the adjusted figures in the third:

$410.17	$.88	$411.05
418.38	.87	419.25
426.74	.87	427.61
435.28	.87	436.15
443.99	.87	444.86
452.87	.87	453.74
461.92	.87	462.79
471.16	.87	472.03
480.58	.87	481.45
490.20	.87	491.07

The following will then be the schedule:

SCHEDULE (F)—ELIMINATION OF RESIDUES; SECOND METHOD

Date	Total Interest 5%	Net Income 4% (—)	Amortization	Book Value	Par
1914, May 1				$104,500.00	$100,000.00
Nov. 1	$ 2,500.00	$ 2,088.95	$ 411.05	104,088.95	
1915, May 1	2,500.00	2,080.75	419.25	103,669.70	
Nov. 1	2,500.00	2,072.39	427.61	103,242.09	
1916, May 1	2,500.00	2,063.85	436.15	102,805.94	
Nov. 1	2,500.00	2,055.14	444.86	102,361.08	
1917, May 1	2,500.00	2,046.26	453.74	101,907.34	
Nov. 1	2,500.00	2,037.21	462.79	101,444.55	
1918, May 1	2,500.00	2,027.97	472.03	100,972.52	
Nov. 1	2,500.00	2,018.55	481.45	100,491.07	
1919, May 1	2,500.00	2,008.93	491.07	100,000.00	
	$25,000.00	$20,500.00	$4,500.00		

In this schedule the income rate varies from 3.99799% to 3.99822%; hence the approximation is sufficiently close for any holdings, except large ones for long maturities.

§ 140. Third Method of Eliminating Residues

For still greater accuracy, we may divide the $8.71 in parts *proportionate* to the amortization. The amortization on the 4% basis amounts to $4,491.29, and we have an extra amount of $8.71 to exhaust. Dividing the latter by the former, we have as the quotient .00194, which is the portion to be added to each dollar of amortization. With this we form a table for the 9 digits:

1 0 0 1 9 4
2 0 0 3 8 8
3 0 0 5 8 2
4 0 0 7 7 6
5 0 0 9 7 0
6 0 1 1 6 4
7 0 1 3 5 8
8 0 1 5 5 2
9 0 1 7 4 6

From this table it is easy to adjust each item of amortization, writing down, for example, to the nearest mill:

410.17	418.38	426.74	435.28
400.776	400.776	400.776	400.776
10.019	10.019	20.039	30.058
.100	8.016	6.012	5.010
.070	.301	.701	.200
410.97	.080	.040	.080
	419.19	427.57	436.12

The respective amounts of amortization, in Schedule (G), vary (at the most) but 8 cents from those shown in Schedule (F).

SCHEDULE (G)—ELIMINATION OF RESIDUES; THIRD METHOD

Date	Total Interest 5%	Net Income 4% (—)	Amortization	Book Value	Par
1914, May 1				$104,500.00	$100,000.00
Nov. 1	$ 2,500.00	$ 2,089.03	$ 410.97	104,089.03	
1915, May 1	2,500.00	2,080.81	419.19	103,669.84	
Nov. 1	2,500.00	2,072.43	427.57	103,242.27	
1916, May 1	2,500.00	2,063.88	436.12	102,806.15	
Nov. 1	2,500.00	2,055.15	444.85	102,361.30	
1917, May 1	2,500.00	2,046.25	453.75	101,907.55	
Nov. 1	2,500.00	2,037.18	462.82	101,444.73	
1918, May 1	2,500.00	2,027.93	472.07	100,972.66	
Nov. 1	2,500.00	2,018.49	481.51	100,491.15	
1919, May 1	2,500.00	2,008.85	491.15	100,000.00	
	$25,000.00	$20,500.00	$4,500.00		

§ 141. Short Terminals

It sometimes happens (though infrequently) that the principal of a bond is payable, not at an interest date, but from one to five months later, making a short terminal period. The following is a very simple method of obtaining the present value in this case. It will not be necessary to demonstrate it, but an example will test it.

Suppose the 5% bond, M N, yielding 4%, bought May 1, 1914, were payable October 1, instead of May 1, 1919, that is, in 10 5/6 periods. The short period is 5/6. The

short ratio (at 4%) will be 1.0166⅔. The short interest (at 5%) will be .02083⅓.

We first ascertain the value for the ten full periods, viz., for $1	1.0449129*
Add to this the short interest	.0208333
	1.0657462
and divide by the short ratio	1.0166667

To perform this division it will be easier to divide 3 times the dividend by 3 times the divisor.

```
3.05 ) 3.1972386 ( Quotient 1.0482750
       3.05
       ----
         1472
         1220
         ----
          2523
          2440
          ----
            838
            610
           ----
            2286
            2135
            ----
             151
             152
            ----
```

Multiplying down by the usual procedure, we have the following schedule:

*See Schedule (A), § 122.

Schedule (H)—Short Terminals

Date	Total Interest 5%	Net Income 4%	Amortization	Book Value	Par
1914, May 1				$104,827.50	$100,000.00
Nov. 1	$ 2,500.00	$ 2,096.55	$ 403.45	104,424.05	
1915, May 1	2,500.00	2,088.48	411.52	104,012.53	
Nov. 1	2,500.00	2,080.25	419.75	103,592.78	
1916, May 1	2,500.00	2,071.86	428.14	103,164.64	
Nov. 1	2,500.00	2,063.29	436.71	102,727.93	
1917, May 1	2,500.00	2,054.56	445.44	102,282.49	
Nov. 1	2,500.00	2,045.65	454.35	101,828.14	
1918, May 1	2,500.00	2,036.56	463.44	101,364.70	
Nov. 1	2,500.00	2,027.29	472.71	100,891.99	
1919, May 1	2,500.00	2,017.84	482.16	100,409.83	
Oct. 1	2,083.33	1,673.50	409.83	100,000.00	
	$27,083.33	$22,255.83	$4,827.50		

§ 142. Rule for Short Terminals

Ascertain the value of the bond for the full number of periods, disregarding the terminal. To this value add the short interest, and divide by the short ratio.

It may be remarked that this same process applies to short initial periods. It even applies to bonds originally issued between interest dates, and also maturing between interest dates; in the case of bonds of this description, the process would be applied twice.

§ 143. Discounting

Hitherto we have calculated the value of the bond at its earliest date, and then obtained the successive values at later dates by multiplication and subtraction. We can also work backwards, however, obtaining each value from the

next later one by addition and division. Let us take, for illustration, the bond shown in Schedule (A), § 122:

Principal to be received at maturity, May 1, 1919	$100,000.00
Coupon to be received at May 1, 1919	2,500.00
Total amount receivable at May 1, 1919	$102,500.00
Discounted value at November 1, 1918, excluding the coupon receivable at that date, found by dividing $102,500.00 by 1.02	$100,490.20
Coupon to be received at November 1, 1918	2,500.00
Total value at November 1, 1918, including the coupon receivable at that date	$102,990.20
Discounted value at May 1, 1918, excluding the coupon receivable at that date, found by dividing $102,990.20 by 1.02	$100,970.78

etc., etc.

In this manner successive terms may be obtained as far as desired.

§ 144. Last Half-Year of Bond

In the last half-year of a bond, its value should be discounted, and not found as in § 128. Thus, if the bond mentioned in § 128 were sold three months prior to maturity, its value would be found by dividing $102,500.00 by 1.01, which would give $101,485.15 "flat," equivalent to $100,-235.15 and interest; whereas by the ordinary rule it would be $100,245.10 (that is, midway between $100,490.20 and $100,000.00). The theoretically exact value (recognizing effective rates, which is never done in business) would be $100,240.13. This is found by multiplying the value six months prior to maturity ($100,490.20) by the square root of 1.02, this root to ten decimal places being 1.0099504938.

The product is $101,490.13, which, less the accrued interest amounting to $1,250.00, gives $100,240.13. To "split the difference" would be an easy way of adjusting the matter, and would be almost exact.

§ 145. Serial Bonds

Bonds are often issued in series so that they mature at various dates. For example, there may be an issue of $30,000.00, of which $1,000.00 is payable after one year, another $1,000.00 after two years, and so on, the final $1,000.00 being payable after thirty years. Other series are more complex, as, for example, $2,000.00 payable each year for five years, and $4,000.00 each year thereafter for ten years. The initial value of a series on any given basis cannot be found by one operation; the initial value of each instalment must first be found, and the sum of these separate initial values gives the initial value of the entire series. After the aggregate initial value has been ascertained, it may, for the purposes of deriving values at succeeding interest periods, be treated as a unit, as if the bonds were not in series. At the end of each of the yearly periods, the ordinary amortization or accumulation would have to be computed, and it would also be necessary to deduct from the total value the par value of the bonds cancelled or retired.

In offering serial bonds for sale, they are often listed as of "average maturity—15½ years." This is entirely delusive, and frequently causes the buyer to believe that he is getting a more favorable basis than will be realized. The only correct valuation of a series is the sum of all its separate values. If we assume that the $30,000.00 above referred to was a series of 5% bonds bought on an assumed 3.50% basis, the true value would be................ $35,005.00
whereas the value for the "average time," i.e.,
15½ years, would be..................... 35,348.22

In computing the rate of income yielded by a series, the income rate corresponding to the average time may be taken as a point of departure, but it will be found that it is invariably too high. For example, let us assume that the thirty 5% serial bonds mentioned above were purchased when first issued at 116.68, and that the income rate thereon is desired.

Looking in Sprague's "Extended Bond Tables," under the 5% bonds in the 15½ year column (average date), we find that the value nearest to 116.68 is......$1,165,200.00 which is at a yield of approximately 3.60%.

If, however, we take off on an adding machine,
the value of a 5% bond due in 30 years at
3.60% $1,255,549.38
the value of a 5% bond due in 29 years at
3.60% 1,250,705.95
the value of a 5% bond due in 28 years at
3.60% 1,245,686.60
and so on, to and including the bond due in
1 year, we shall find that the total value of
the 30 bonds is........................34,531,390.28
and that the average price is therefore 115.10.

An income yield of 3.55% for the above bonds will result in a value of 116.06, while a yield of 3.50% will disclose a value of 116.68. The true yield on these bonds, if bought at 116.68, is therefore not 3.60%, as seems at first apparent, but is 3.50%.

§ 146. Irredeemable Bonds

Sometimes, as in the case of British Consols, there is no right nor obligation of redemption. If the government wishes to pay off any of its bonds, it has to buy them at the market price. With this class of bonds, there is no question of amortization; the investment is simply a perpetual an-

nuity. The cash interest is all revenue, and the original cost is the constant book value. If £100 of 4% Consols be bought at 96, the income is £4 per annum, and the book value is £96. Since the investment of £96 produces £4 annually, the rate of income is 4 ÷ 96, or 4 1/6%.

§ 147. Optional Redemption

Sometimes the issuer of a bond has the *right* to redeem at a certain date earlier than the date at which he *must* redeem. It must always be expected that this right will be exercised if profitable to the issuer; hence, to be conservative, a purchaser, when buying this class of bonds at a premium, must always consider them as maturing, or reaching par, at the earlier date. On the other hand, bonds of this character bought at a discount must be considered as running to the longer date. If the bonds bought at a premium run to the very latest date, or if the bonds bought at a discount are called for redemption at an earlier date than was anticipated by the investor, he will, in either case, receive a higher yield in his income rate than he would have received on the more conservative basis. The element of chance enters in here, but, to be safe, the purchaser should always consider that the chances will go against him; he will then have all to gain and nothing to lose.

The option of redemption is sometimes attended by a premium. For example, the issuer of a thirty-year bond reserves the right to redeem after twenty years at 105. Where bonds are bought at such an income yield that after twenty years the book value will be more than 105, the right of redemption at 105 is a detriment to the purchaser. In such a case as this, the safe and conservative purchaser should buy at such an income basis as will bring the book value at 105 or below at the end of twenty years.

There is also a form of bond issue, not uncommon in

Europe, where a certain or indefinite number of bonds is drawn by lot each year for the purposes of retirement. As these bonds are usually issued at a discount, those which are drawn at the earlier dates are the more profitable. The investor, however, in estimating his income, must assume that his particular bonds will be among the last ones drawn. If drawn at earlier dates, there is a profit exactly the same as that arising from a sale above book value.

§ 148. Bonds as Trust Fund Investments

A bond which has been purchased by a trustee at a premium is subject to amortization in the absence of testamentary instructions to the contrary. The trustee has no right to pay over the full cash interest to the life tenant, because he must keep the principal intact for the remainder man. If, for example, the trustee were to invest $104,491.29 in a 5% bond having five years to run, and if he were to pay over the full amount of the coupons to the life tenant for the period of five years, the fund at the end of the period would simply be the par value of the bonds, $100,000.00, and would therefore be depleted to the extent of $4,491.29, to the manifest injury of the remainder man. Since the investment is on a 4% basis, the trustee should pay over at the end of the first half-year only 2% of $104,491.29 (or $2,089.83), and not 2½% of $100,000.00 (or $2,500.00). He then has $410.17 cash to reinvest, and the fund, including this, is still $104,491.29. It may be difficult to invest the $410.17 at as favorable a rate as the bonds, very small and very large amounts being most difficult to invest. The trustee can deposit it in a trust company, at least, and receive interest at some rate, however small.

§ 149. Payments to Life Tenant

At the end of the second half-year, the net income on

the bond is only \$2,081.62; but, in addition to this amount, the life tenant is also entitled to the interest on the \$410.17. If this has been reinvested at exactly 4% (interest payable semi-annually), the interest thereon is \$8.20, and the total amount payable to the life tenant is \$2,081.62 + \$8.20 = \$2,089.82. This is practically the same amount of income as in the first half-year. The difference between the coupons received (\$2,500.00) and the net income (\$2,089.82) is the amortization (\$418.38), which is deposited or invested as before. The trustee now has in the fund:

Book value of the bonds	\$103,662.74
Invested in Trust Company or otherwise at end of first half-year	410.17
Invested in Trust Company or otherwise at end of second half-year	418.38
Total	\$104,491.29

He has paid over all of the new interest earned, and he has kept the corpus or principal intact.

§ 150. Effect of Varying Rates on Investments

Suppose, however, that the trustee was not able to get 4% on the \$410.17, but only 3%, so that from this source would come only \$6.15, making the total income \$2,081.62 plus \$6.15, or \$2,087.77. This shows a slight falling off in income, but that is to be expected when part of an investment is returned and reinvested at a lower rate. If the reinvestment had been at 4½%, the income would have been \$2,090.85, slightly more than the first half-year, owing to the improved demand for capital. It might be urged that the life tenant ought to receive \$2,089.83 semi-annually—no more, no less—being at a 4% rate on \$104,491.29. This would leave \$410.17 each half-year to be invested in a

sinking fund, from which no interest should be drawn, but which should be left to accumulate to maturity, when it would exactly replace the premium, *if compounded at 4%.* But this hope might not be realized. Very likely the average rate would be less or more than 4% and not exactly 4%. If less, the original fund would be to some extent depleted, and the remainder man wronged; if more, there would be too much in the fund, and the life tenant would receive too little. It seems, therefore, that the sinking fund principle is not correct in a case like this, and that, at all events, the original fund should be kept constant, neither increased nor diminished. So much of the semi-annual receipts as are not necessary to maintain the constancy of the fund due to the remainder man should be paid over as income to the life tenant.

§ 151. Example of Payments to Life Tenant

A two-year, 4% bond, par value $10,000, bought at $10,192.72, would be scheduled thus:

Coupon	Income	Cash	Bond
			$10,192.72
$200.00	$152.89	$47.11	10,145.61
200.00	152.18	47.82	10,097.79
200.00	151.47	48.53	10,049.26
200.00	150.74	49.26	10,000.00
$800.00	$607.28	$192.72	

The life tenant would receive, at the end of the first half-year, $152.89; at the end of the second, $152.18 + whatever the $47.11 cash had earned; at the end of the third, $151.47 + whatever $94.93 had earned; at the close, $150.74 + whatever $143.46 had earned. If the cash bal-

ance were periodically deposited in a trust company at 3% (payable semi-annually), the life tenant would receive a uniform income of $152.89.

§ 152. Cullen Decision

In a New York case (38 App. Div. 419), Justice Cullen very clearly lays down the law as to the duty of the trustee to reserve a part of the interest to provide for the premium, and says that "any other view would lead to the impairment of the principal of the trust, to protect the integrity of which has always been the cardinal rule of courts of equity." He says further: "If one buys a ten-year five per cent bond at one hundred and twenty, the true income or interest the bond pays is not 4 1/6% on the amount invested, nor 5% on the face of the bond, but 2 7/10% on the investment, or 3 24/100% on the face of the bond. The matter is simply one of arithmetical calculation, and tables are readily accessible, showing the result of the computation."

§ 153. Cullen Decision Scheduled

Consulting one of the tables referred to by Justice Cullen (Sprague's "Extended Bond Tables"), and looking in the 5% tables under the column headed "10 Years," we find that the value nearest to $120,000.00 is $120,038.997, which is opposite the net income rate of 2.70%. As stated by Justice Cullen, the bonds, while bearing a coupon rate of 5%, actually net, therefore, only 2.7% on account of the high premium. With a slight correction in the initial figures in order to make the income rate exactly 2.7%, and assuming a par value of $100,000.00, the illustration as given in the above case by Justice Cullen, when tabulated to show the present value, the income, and the amount reinvested, would work out as follows:

Total Interest	Income Paid Over	Reinvested	Present Value
			$120,039.00
$2,500.00	$1,620.53	$879.47	119,159.53
2,500.00	1,608.65	891.35	118,268.18
2,500.00	1,596.62	903.38	117,364.80
2,500.00	1,584.42	915.58	116,449.22
2,500.00	1,572.07	927.93	115,521.29
2,500.00	1,559.53	940.47	114,580.82
2,500.00	1,546.85	953.15	113,627.67
2,500.00	1,533.97	966.03	112,661.64
2,500.00	1,520.93	979.07	111,682.57
2,500.00	1,507.72	992.28	110,690.29
2,500.00	1,494.31	1,005.69	109,684.60
2,500.00	1,480.75	1,019.25	108,665.35
2,500.00	1,466.93	1,033.07	107,632.28
2,500.00	1,453.08	1,046.92	106,585.36
2,500.00	1,438.91	1,061.09	105,524.27
2,500.00	1,424.57	1,075.43	104,448.84
2,500.00	1,410.06	1,089.94	103,358.90
2,500.00	1,395.35	1,104.65	102,254.25
2,500.00	1,380.43	1,119.57	101,134.68
2,500.00	1,365.32	1,134.68	100,000.00
$50,000.00	$29,961.00	$20,039.00	

§ 154. Unjust Feature of Cullen Decision

The foregoing schedule is perfectly correct, but we can scarcely agree with the method described further on in the same opinion, as follows: "There is, however, a simpler way of preserving the principal intact—the method adopted by the learned referee. He divided the premium paid for the bonds by the number of interest payments, which would

be made up to the maturity of the bonds, and held that the quotient should be deducted from each interest payment and held as principal. These deductions being principal, the life tenant would get the benefit of any interest that they might earn. We do not see why this plan does not work equal justice between the parties." The reason "why it does not work equal justice" is that the life tenant in the earlier years receives much less than his due share of the income, but from year to year he gradually receives more and more, until he receives more than his share; but not until the very last payment does he overtake his true share. Thus, if he dies before the maturity of the bonds, it is certain that "equal justice" will not have been done, and that the remainder man will have had altogether the best of it.

The schedule under the referee's plan would work out as follows:

Total Interest	Income Paid Over	Reinvested	Present Value
			$120,039.00
$2,500.00	$1,498.05	$1,001.95	119,037.05
2,500.00	1,498.05	1,001.95	118,035.10
2,500.00	1,498.05	1,001.95	117,033.15
2,500.00	1,498.05	1,001.95	116,031.20
2,500.00	1,498.05	1,001.95	115,029.25
2,500.00	1,498.05	1,001.95	114,027.30
2,500.00	1,498.05	1,001.95	113,025.35
2,500.00	1,498.05	1,001.95	112,023.40
2,500.00	1,498.05	1,001.95	111,021.45
2,500.00	1,498.05	1,001.95	110,019.50
etc.	etc.	etc.	etc.

Assuming that, under each plan, the reinvested funds would earn the same rate of income as the original invest-

ment (i.e., 2.7%), the total semi-annual income of the remainder man would be as follows:

Under Plan in § 153	Under Plan in § 154
$1,620.53	$1,498.05
1,620.53	1,511.58
1,620.53	1,525.10
1,620.53	1,538.63
1,620.53	1,552.16
1,620.53	1,565.68
1,620.53	1,579.21
1,620.53	1,592.73
1,620.53	1,606.26
1,620.53	1,619.79
etc.	etc.

A comparison of the two columns will show the injustice of the referee's plan toward the life tenant, and substantiate the equity of the plan of scientific amortization set forth in the schedule in § 153.

§ 155. Bond Tables

Mention has been made heretofore of bond tables. These tables show the values of bonds at various coupon rates, yielding various rates of net income, and due in different periods from one-half year to one hundred years. The usual tables refer to bonds whose coupons are payable semi-annually, but there are generally supplementary tables by the use of which values of those bonds may be ascertained whose coupons are payable annually or quarterly. The following table is taken from page 80 of Sprague's "Extended Bond Tables," and sets forth (in part) the values of the bond mentioned in Schedule (A), § 122:

Bond Table

Values, to the Nearest Cent, of a Bond for $1,000,000 at 5% Interest, Payable Semi-Annually

Net Income	3 Years	3½ Years	4 Years	4½ Years	5 Years
2.50	1,071,825.12	1,083,284.07	1,094,601.55	1,105,779.31	1,116,819.07
2.55	1,070,328.46	1,081,538.84	1,092,608.09	1,103,537.98	1,114,330.27
2.60	1,068,834.33	1,079,796.97	1,090,618.92	1,101,302.00	1,111,847.97
2.65	1,067,342.73	1,078,058.45	1,088,634.05	1,099,071.36	1,109,372.18
2.70	1,065,853.65	1,076,323.28	1,086,653.46	1,096,846.04	1,106,902.85
2.75	1,064,367.09	1,074,591.45	1,084,677.14	1,094,626.03	1,104,439.98
2.80	1,062,883.04	1,072,862.96	1,082,705.08	1,092,411.33	1,101,983.56
2.85	1,061,401.50	1,071,137.78	1,080,737.28	1,090,201.90	1,099,533.55
2.90	1,059,922.46	1,069,415.93	1,078,773.71	1,087,997.74	1,097,089.94
2.95	1,058,445.92	1,067,697.38	1,076,814.37	1,085,798.84	1,094,652.71
3.00	1,056,971.87	1,065,982.14	1,074,859.25	1,083,605.17	1,092,221.85
3.05	1,055,500.31	1,064,270.19	1,072,908.34	1,081,416.74	1,089,797.33
3.10	1,054,031.24	1,062,561.54	1,070,961.63	1,079,233.51	1,087,379.13
3.15	1,052,564.64	1,060,856.16	1,069,019.11	1,077,055.49	1,084,967.25
3.20	1,051,100.52	1,059,154.06	1,067,080.77	1,074,882.64	1,082,561.66
3.25	1,049,638.87	1,057,455.22	1,065,146.59	1,072,714.97	1,080,162.34
3.30	1,048,179.68	1,055,759.65	1,063,216.58	1,070,552.46	1,077,769.27
3.35	1,046,722.96	1,054,067.33	1,061,290.71	1,068,395.09	1,075,382.44
3.40	1,045,268.68	1,052,378.25	1,059,368.98	1,066,242.85	1,073,001.82
3.45	1,043,816.86	1,050,692.42	1,057,451.38	1,064,095.73	1,070,627.41
3.50	1,042,367.48	1,049,009.81	1,055,537.90	1,061,953.71	1,068,259.17
3.55	1,040,920.54	1,047,330.43	1,053,628.52	1,059,816.78	1,065,897.10
3.60	1,039,476.04	1,045,654.27	1,051,723.25	1,057,684.92	1,063,541.18
3.65	1,038,033.97	1,043,981.31	1,049,822.06	1,055,558.13	1,061,191.38
3.70	1,036,594.33	1,042,311.57	1,047,924.95	1,053,436.38	1,058,847.70
3.75	1,035,157.11	1,040,645.01	1,046,031.91	1,051,319.67	1,056,510.11
3.80	1,033,722.30	1,038,981.65	1,044,142.93	1,049,207.98	1,054,178.59
3.85	1,032,289.91	1,037,321.47	1,042,258.00	1,047,101.30	1,051,853.13
3.90	1,030,859.92	1,035,664.46	1,040,377.11	1,044,999.62	1,049,533.71
3.95	1,029,432.34	1,034,010.63	1,038,500.25	1,042,902.92	1,047,220.32
4.00	1,028,007.15	1,032,359.96	1,036,627.41	1,040,811.18	1,044,912.93
4.05	1,026,584.36	1,030,712.44	1,034,758.58	1,038,724.41	1,042,611.52
4.10	1,025,163.96	1,029,068.07	1,032,893.74	1,036,642.57	1,040,316.09
4.15	1,023,745.94	1,027,426.84	1,031,032.90	1,034,565.67	1,038,026.61
4.20	1,022,330.31	1,025,788.74	1,029,176.04	1,032,493.68	1,035,743.07
4.25	1,020,917.04	1,024,153.77	1,027,323.16	1,030,426.59	1,033,465.45

Net Income	3 Years	3½ Years	4 Years	4½ Years	5 Years
4.30	1,019,506.15	1,022,521.93	1,025,474.23	1,028,364.40	1,031,193.73
4.35	1,018,097.62	1,020,893.20	1,023,629.26	1,026,307.08	1,028,927.90
4.40	1,016,691.46	1,019,267.57	1,021,788.23	1,024,254.63	1,026,667.93
4.45	1,015,287.65	1,017,645.05	1,019,951.13	1,022,207.03	1,024,413.82
4.50	1,013,886.19	1,016,025.62	1,018,117.96	1,020,164.27	1,022,165.54
4.55	1,012,487.08	1,014,409.27	1,016,288.70	1,018,126.33	1,019,923.08
4.60	1,011,090.32	1,012,796.01	1,014,463.35	1,016,093.21	1,017,686.42
4.65	1,009,695.89	1,011,185.82	1,012,641.90	1,014,064.89	1,015,455.55
4.70	1,008,303.80	1,009,578.70	1,010,824.33	1,012,041.35	1,013,230.44
4.75	1,006,914.03	1,007,974.64	1,009,010.63	1,010,022.60	1,011,011.08
4.80	1,005,526.59	1,006,373.63	1,007,200.81	1,008,008.60	1,008,797.46
4.85	1,004,141.48	1,004,775.67	1,005,394.84	1,005,999.36	1,006,589.56
4.90	1,002,758.67	1,003,180.75	1,003,592.72	1,003,994.85	1,004,387.36
4.95	1,001,378.18	1,001,588.86	1,001,794.45	1,001,995.07	1,002,190.85
5.00	1,000,000.00	1,000,000.00	1,000,000.00	1,000,000.00	1,000,000.00

§ 156. Features of the Bond Table

The relations existing between succeeding values on the same horizontal line of the table are readily seen, these values being computed at the same income rate but for different semi-annual periods. For example, on a 2.50% basis, the value of this 5% bond 5 years prior to maturity is $1,116,819.07. At 4½ years prior to maturity, its value is 1.0125 (which is the semi-annual income rate) times $1,-116,819.07, producing $1,130,779.31, from which must be deducted the semi-annual coupons ($25,000.00), giving as a final result $1,105,779.31. The net income yields shown in the preceding table vary to the extent of .05%. Usually there are supplementary tables by the use of which values of bonds may be calculated at different income yields differing by only .01%. In this manner, the values of the bond shown in the preceding table may be computed on an income yield of 2.51%, 2.52%, 2.53%, etc.

CHAPTER XII

SUMMARY OF COMPOUND INTEREST PROCESSES

§ 157. Rules and Formulas

In the present chapter are given in condensed and symbolic form the rules and formulas which have been explained in the preceding chapters.

§ 158. Rules

(1) To find the ratio of increase:
  Add 1 to the rate of interest.

(2) To find the amount of $1:
  Multiply 1 by the ratio as many times as there are periods.

(3) To find the present worth of $1, or to discount $1:
  Divide 1 by the ratio as many times as there are periods.

(4) To find the total interest on $1:
  Subtract 1 from the amount.

(5) To find the total discount on $1:
  Subtract the present worth from 1.

(6) To find the amount of an annuity of $1:
  Divide the total interest by the rate of interest.

(7) To find the present worth of an annuity of $1:
  Divide the total discount by the rate of interest.

(8) To find the rent of an annuity worth \$1, or what annuity can be bought for \$1:

Divide 1 by the present worth of the annuity.

(9) To find what annuity (sinking fund) will produce \$1:

Divide 1 by the amount of the annuity.

(10) To find the premium on a bond, or the discount on a bond:

Consider the difference between the cash and income rates as an annuity to be valued, and find its present worth at the income rate.

(11) To find the value of a bond:

In case the cash rate is greater than the income rate, the bond is at a premium; therefore, add par to the premium.

In case the income rate is greater than the cash rate, the bond is at a discount; therefore, subtract the discount from par.

§ 159. Formulas

i is the rate of interest or the interest on unity for 1 period. n is the number of periods. c is the cash rate on a bond.

(1) Ratio of Increase $= 1+i$

(2) Amount $= (1+i)^n$

(3) Present Worth or Disc. $= \dfrac{1}{(1+i)^n}$

(4) Total Interest $= (1+i)^n - 1$

(5) Total Discount $= 1 - \dfrac{1}{(1+i)^n}$

(6) Amount of Annuity $= \dfrac{(1+i)^n - 1}{i}$

(7) Present Worth of Annuity $= \dfrac{1 - \dfrac{1}{(1+i)^n}}{i}$

(8) Rent of Annuity $= \dfrac{i}{1 - \dfrac{1}{(1+i)^n}}$

(9) Sinking Fund $= \dfrac{i}{(1+i)^n - 1}$

(10) Premium on Bond $= \dfrac{c-i}{i}\left(1 - \dfrac{1}{(1+i)^n}\right)$

(11) Discount on Bond $= \dfrac{i-c}{i}\left(1 - \dfrac{1}{(1+i)^n}\right)$

(12) Value of Bond (at a Premium) $= 1 + \dfrac{c-i}{i}\left(1 - \dfrac{1}{(1+i)^n}\right)$

(13) Value of Bond (at a Discount)* $= 1 - \dfrac{i-c}{i}\left(1 - \dfrac{1}{(1+i)^n}\right)$

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* This formula is equivalent to Formula (12).

CHAPTER XIII

ACCOUNTS—GENERAL PRINCIPLES

§ 160. Relation of General Ledger to Subordinate Ledgers

In any extensive system of accountancy, in order to fulfill the opposite requirements of minuteness and comprehensiveness, it is necessary to keep, in some form, a general ledger and various subordinate ledgers. Each account in the general ledger, as a rule, comprises or summarizes the entire contents of one subordinate ledger. Each account of the general ledger comprises groups of similar accounts, which are handled in the subordinate ledgers as individual assets or as groups which may be treated as individuals. It is the province of the general ledger to give information in grand totals as an indicator of tendencies; while the function of the subordinate ledger is to give all desired information as to details, even beyond the figures required for balancing—facts not only of numerical accountancy, but descriptive, cautionary, or auxiliary. Thus the general ledger may contain a "Mortgages" account, which will show the increase or decrease of the amount invested on mortgage, and the resultant or present amount; the mortgage ledger will contain an account for each separate mortgage, with additional information as to interest, taxes, insurance, title, ownership, security, valuation, or any other thing useful or necessary to be known.

§ 161. The Interest Account

We shall assume that a general ledger exists with subordinate or class ledgers. We shall also assume that the

accounts are to be so arranged as to give currently the amount of interest earned up to any time, and the amount outstanding and overdue at any time. It would hardly seem necessary to argue this point, were it not that many large investors pay no attention to interest until it matures, and some do not carry it into their accounts until it is paid. They are compelled to make an adjustment on their periodical balancing dates "in the air," compiling it from various sources without check, which seems as crude as it would be to take no account of cash, except by counting it occasionally. The Profit and Loss account depends for its accuracy upon the interest earned, not upon the interest falling due, nor upon the interest collected; and the accruing of interest is a fact which should be recognized and recorded.

§ 162. Mortgage and Loan Accounts

In considering the forms of account for investments, we will first take up, as being simpler, those in which there is never any value to be considered other than par, such as mortgages and loans upon collateral security. Both of these classes of investments are for comparatively short terms, and are usually the result of direct negotiation between borrower and lender, and not the subject of purchase and sale; hence, changes in rate of interest are readily effected by agreement, and do not result in a premium or discount.

CHAPTER XIV

REAL ESTATE MORTGAGES

§ 163. Nature of Loans on Bond and Mortgage

The instruments which we have spoken of as "bonds" are very often secured by a mortgage of property. But one mortgage will secure a great number of bonds, the mortgagee being a trustee for all the bondholders. In contrast with these instruments, those of which we now speak are the ordinary "bond and mortgage," by which the investor receives from the borrower two instruments: the one an agreement to pay, and the other conferring the right, in case default is made, to have certain real estate sold, and the proceeds used to pay the debt. As only a portion of the value of the real estate is loaned, the reliance is primarily on the mortgage rather than on the bond. Therefore, the mortgagee must be vigilant in seeing that his margin is not reduced to a hazardous point. This may happen by the depreciation of the land for various economic reasons; by the deterioration of the structures thereon through time or neglect; by destruction through fire; or by the non-payment of taxes, which are a lien superior to all mortgages. By reason of these risks a mortgage loan is seldom made for more than a few years; but after the date of maturity, extensions are made from time to time; or, even more frequently, without formal extension, the loan is allowed to remain "on demand," either party having the right to terminate the relation at will. A large proportion of outstanding mortgages are thus "on sufferance," or payable on

demand. The market rate of interest seldom causes the obligation to change hands at either a premium or a discount; hence we may ignore that feature, referring the exceptional cases, where it occurs, to the analogy of bonds.

The two instruments, bond and mortgage, relate to the same transaction, are held by the same owner, and for most purposes are treated as a unit. In bookkeeping, the investment must likewise be treated as a unit, both as to principal and income.

§ 164. Separate Accounts for Principal and Interest

It is desirable to know at any time how much is due on principal, allowing for any partial payments. It is also desirable to know what interest, if any, is due and payable, and to be able to look after its collection. An account with principal and an account with interest are therefore requisite. It is better, however, if these two accounts for the same mortgage be adjacent.

§ 165. Interest Debits and Credits

Accrued interest need not be considered as to each mortgage. It should be treated in bulk, in the same manner as the revenue of the aggregate mortgages, as will be explained hereafter. The Interest account (on the investor's books) here referred to is debited on the day when the interest becomes a matured obligation, and credited when that obligation is discharged.

§ 166. Characteristics of Modern Ledger

Those who adhere to the original form of the Italian ledger will probably be averse to combining with the ledger account any general business information; in fact, that form is not suited for such purposes, and is not adapted to containing anything but the bare figures that will make the

trial balance prove. But the modern conception of a ledger is broader and more practical: it should be an encyclopedia of information bearing on the subject of the account; it should be specialized for every class ledger; it should be of any form which will best serve its purposes, regardless of custom or tradition.

§ 167. The Mortgage Ledger

The form of mortgage ledger which seems best to the author contains four parts:

(1) Descriptive.
(2) Account with principal.
(3) Account with interest.
(4) Auxiliary information.

These may occupy four successive pages, or two pages, if preferred. In the latter case, if kept in a bound volume, the arrangement whereby two of these parts are on the left-hand page and two on the right, confronting each other, is a convenient one, giving all the facts at one view. For a loose-leaf ledger, the order (1), (2), (3), (4) will generally be found the best.

§ 168. Identification of Mortgages by Number

Mortgages should be numbered in chronological order, and every page or document should bear the number of the mortgage loan to which it refers.

§ 169. The "Principal" Account

The account with principal may be in the ordinary ledger form; but what is known as the balance-column, or three-column form, will be found more convenient. It contains but one date column, so that successive transactions, whether payments on account, or additional sums loaned, appear in their proper chronological order.

§ 170. Special Columns for Mortgagee's Disbursements

The mortgage usually contains clauses which permit the mortgagee, when the mortgagor fails to make any necessary payment for the benefit of the property, like taxes and insurance premiums, to step in and advance the money, which he has the right to recover with interest. It will be useful to have columns for these disbursements and the corresponding reimbursements (§ 179, Form II).

§ 171. The Interest Account

The Interest account (§ 179, Form III) may be very simple. It contains two columns, one for debits on the day when interest falls due, the other for crediting when it is collected. The entries in the Interest account will naturally be much more numerous than in the Principal account; hence, this pair of columns may be repeated several times. The arrangement shown has been found advantageous.

§ 172. Interest Due

Experience has shown that the safest way to insure attention to the punctual and accurate collection of interest is to charge up, systematically, under the due date, every item, and to let it stand as a debit balance until collected. Many attempt to accomplish the same purpose by merely marking "paid" on a list; but this is apt to lead to confusion, and it is difficult to verify afterward the state of the accounts on any given date.

§ 173. Books Auxiliary to Ledger

It is not proposed in this treatise to prescribe the forms of posting mediums (cash book, journal, etc.) from which the postings in the ledger are made, because these forms are so largely dependent upon the peculiarities of the

business, and have deviated so far from the traditional Italian form, that no universal type could be presented. We shall, however, give the debit and credit formulas underlying the postings, and will suggest auxiliary books or lists for making up the entries.

§ 174. The "Due" Column

The formula for the "Due" column of the Interest account is:

Interest Due / Interest Accrued $.........

It is a transfer from one branch of interest receivable, viz., that which is a debt, but not yet enforceable, to another branch, viz., that which is a matured claim.

§ 175. Interest Account Must Be Analyzed

In the general ledger the entry will be simply as above:

Interest Due / Interest Accrued $.........

and this may be a daily, weekly, or monthly entry, or for any other space of time, according to the general practice of the business; the monthly period is most in use, and we shall take that as the standard. The credit side of the entry (/ Interest Accrued) is not regarded in the subordinate ledger (§ 160), but the debit entry (Interest Due /) must be somewhere analyzed into its component parts; in other words, there must be somewhere a list, the total of which is the aggregate falling due on *all* mortgages, and the items of which are the interest falling due on *each* mortgage.

§ 176. Form of "Interest Due" Account

The following heading will suggest the requirements for such a list, the form to be modified to conform to the general system.

REGISTER OF INTEREST DUE

Mortgages						
Date	No.	Principal	Rate	Time	Interest	Total

§ 177. Forms for Mortgage Account

Form I (§ 179) of the Mortgage account is descriptive. Its elements may be placed in various orders of arrangement. Form IV (§ 179) combines all the particulars ordinarily required in the State of New York.

§ 178. Loose-Leaf and Card Records

Form IV (§ 179) is not an essential feature of mortgage loan accounts, and may be replaced by card lists, if preferred. Yet, if there is space, there are advantages in having all the information about a certain mortgage accessible at one time, and concentrated in one place. The changing names and addresses of the mortgagors and owners, and the successive policies of insurance require for their record considerable space, which may be conveniently arranged under the headings in Form IV.

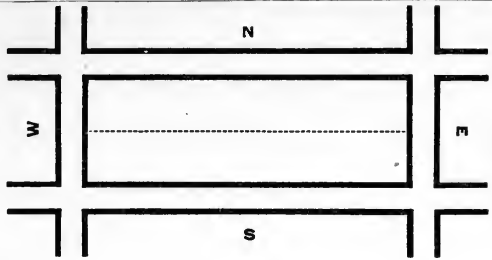
The card form of mortgage ledger is very convenient in many respects, and the forms here given may be rearranged to suit different sizes of cards. Both in cards and loose leaves it will be helpful to use different colors for pages of different contents. Where interest on different mortgages fall due in different months, tags marked "J J," "F A," "M S," "A O," "M N," and "J D," may project from the interest-sheet like an index, the tags of each month at the same distance from the top. This will greatly facilitate the compiling of the register of interest due.

§ 179. Forms of Mortgage Loan Accounts

FORM I

MORTGAGE ON PROPERTY SITUATED No.

Section No..................
Ward No.............. ....
Block No. ....
Lot No. ....
Map No...................
Recorded..............19..
Liber No..................
Page No...................



VALUATIONS			
Date	Land	Improvements	Total

Attorney's No............. Discharged..............19.. by................

Improv. ments

Bond of
Mortgage also signed by
Assignments

Executed 19
Due 19

Title Policy No.

Additional Papers

Application accepted 19 , Minute Book No. , page Rate of Interest,

TAXES PAID				
1914	1917	1920	1923	1926
1915	1918	1921	1924	1927
1916	1919	1922	1925	1928

REMARKS OR MEMORANDA

FORM II

No. LOAN TO

DATE	PRINCIPAL			INSURANCE PREMIUMS, TAXES AND OTHER DISBURSEMENTS	
	LOANED	PAID IN	BALANCE	ADVANCED	REFUNDED

FORM III

INTEREST ACCOUNT, MORTGAGE No.

RATE

DATE		DUE	PAID	DATE		DUE	PAID	DATE		DUE	PAID

Form IV

No. LOAN TO

Addresses for Sending Notices			Insurance				
DATE	OWNER OR AGENT	ADDRESS	COMPANY	POLICY NO.	AMOUNT	EXPIRATION	RENEWED TO

§ 180. Reverse Posting of Interest Register

The interest register should always be made up and proved (subjected to modifications) in advance. In doing this, instead of making the computations in the register and posting thence to the ledger, a surer way is by "reverse posting"; that is, making the computation from the data in the ledger and entering it there at once, in pencil if preferred; then copying the items into the register, where the total can be proved. When this has been done, we can be sure without further check that the ledger is correct. (See also § 183.)

§ 181. Handling Receipts and Notices

It is desirable, also, to have receipts prepared in advance ready for signature. The correctness of these receipts may be assured by introducing them into the "reverse posting" process, as follows: Having made the computation on the ledger, prepare the receipts from the ledger, copying down the figures just as they appear; from the receipts make up the register, which prove as before. This method may be extended to the notices, if any are sent to the mortgagors, the notice being derived from ledger account, the receipt from the notice, and the register from the receipt; if the register proves correct, the correctness of its antecedents is established. These interest notices may be made of assistance in the bookkeeping, if their return is insisted upon and made convenient. Below the formal notification of the sum falling due on such a date, with all particulars, is a blank form somewhat as follows:

In payment of the above interest I inclose check on the .... for $................ and request you to acknowledge receipt as per the address below.

Signature ....

Address ....

The notice upon its being received, together with the check, becomes a "voucher-with-cash," and the entries on the cash book and the interest page of the mortgage ledger are made directly from the documents. Book-to-book posting, which formerly was the only method of rearranging items, is becoming obsolete, being superseded in many businesses by voucher or document posting. By the carbon process the notice and the receipt may be filled in simultaneously in fac-simile.

§ 182. Mortgages Account in General Ledger

The class account "Mortgages" in the general ledger is simply kept to show aggregates. Its entries are, as far as possible, monthly, the posting mediums being so arranged as to give a monthly total of the same items which have already been posted in detail to the mortgage ledger. The standard form of ledger account may be used, or the three column. In the former, the debits and credits of the same month should be kept in line, even though one line of paper be wasted.

(Standard Form)

MORTGAGES

1914					1914				
Jan.	0	Balance	$169,000	00					
"	1-31	Total loaned	12,000	00	Jan.	1-31	Total paid in	$ 7,000	00
Feb.	1-28	" "	10,000	00					
March	1-31	" "	50,000	00	March	1-31	" " "	32,000	00
April	1-30	" "	20,000	00	April	1-30	" " "	40,000	00
May	1-31	" "	5,000	00	May	1-31	" " "	12,000	00
June	1-30	" "	10,000	00	June	1-30	" " "	3,000	00
					"	30	Balance	182,000	00
			$276,000	00				$276,000	00
July	0	Balance	$182,000	00					

(Three-Column Form)

MORTGAGES

1914			Dr.		Cr.		Balance	
Jan.	0						$169,000	00
"	— —	Transactions for month	$ 12,000	00	$ 7,000	00	174,000	00
Feb.	— —	" " "	10,000	00			184,000	00
March	— —	" " "	50,000	00	32,000	00	202,000	00
April	— —	" " "	20,000	00	40,000	00	182,000	00
May	— —	" " "	5,000	00	12,000	00	175,000	00
June	— —	" " "	10,000	00	3,000	00	182,000	00
		Transactions for half-year	$107,000	00	$94,000	00	+ $ 13,000	00
July	0						$182,000	00

§ 183. Tabular Register

In order to keep the fullest control of the interest accruing and falling due periodically, it is useful to keep tabular registers, classifying the mortgages, first, by rates of interest; and second, by the months in which the interest comes due. Those investors who require all interest to be paid at the same date can dispense with the latter. The two presentations or *developments* may be on opposite pages, both proved by the same totals.

FORM I

MORTGAGES CLASSIFIED BY RATES OF INTEREST

Date	Total	Changes	3½%	4%	4½%	5%	6%
1914							
Jan. 0	$169,000		$11,000	$43,000	$50,000	$60,000	$5,000
	7,000	262 –		7,000			
	$162,000						
	12,000	984 +			12,000		
Feb. 0	$174,000		$11,000	$36,000	$62,000	$60,000	$5,000

FORM II

MORTGAGES CLASSIFIED BY INTEREST DATES

Date	Total	Changes	J J	F A	M S	A O	M N	J D
1914								
Jan. 0	$169,000		$23,000	$30,000	$4,000	$8,000	$90,000	$14,000
	7,000	262 –						7,000
	$162,000							
	12,000	984 +		12,000				
Feb. 0	$174,000		$23,000	$42,000	$4,000	$8,000	$90,000	$ 7,000

The numbers in the column headed "Changes" are the serial numbers of the respective mortgages.

§ 184. Equal Instalment Method

Mortgages payable in equal instalments, each covering the interest and part of the principal, present no special difficulty. The value of the periodical instalment should first be ascertained, as shown in § 76; then it should be separated by means of a schedule into "Interest on Balances" and "Payments on Principal," down to the final payment.

CHAPTER XV

LOANS ON COLLATERAL

§ 185. Short Time Loans on Personal Property

Short time investments are often made upon the security or pledge of bonds, stocks, goods, or other personal property valued at more than the amount of the loan. Frequently these are payable on demand, and are known as "call loans." It is evident that the rate of interest may be readjusted every day, or as often as either party is dissatisfied, and, if an agreement cannot be reached, the loan will be paid off. Hence, neither premium nor discount will occur in this kind of investment, and, as in the case of mortgages, we need only concern ourselves with principal (at par) and interest.

§ 186. Forms for Loan Accounts

The accountancy of loans is even simpler than that of mortgages, and it is only necessary to give two forms, one for Principal account and Interest account, and the other for the "Register of Collateral" (§ 188). The latter account, at least, is often kept on cards or on envelopes, and there is great danger of the history becoming confused and unintelligible through erasures and changes in the amounts of collateral, when substitutions are made. When part of a certain security is withdrawn, the entire line should be ruled out, and the reduced quantity rewritten on a new line. When a card becomes at all complicated, it is better to insert a fresh one, rewriting all collateral, but keeping the former card with the new one until the loan is entirely liquidated.

§ 187. Requirements for Interest Account

The Interest account may be kept concurrent with the Principal account—that is, using up the same number of lines in each. In the suggested form there is a column for interest accrued as well as for interest due. The interest accrued column is merely a preparatory calculation column, entered up at each change of rate or principal, so that there may be only one computation to make when the interest becomes due. With this exception, the mechanism of the loan ledger is practically the same as that of the mortgage ledger, and the general ledger account of loans will be similar to that of mortgages.

As the principal and the interest in bond accounts are so intimately connected, it will be advisable to consider the accounting of interest revenue more fully before taking up the subject of bond accounts. This is done in the following chapter.

§ 188. Forms for Collateral Loan Accounts

On the following page are shown two suggested forms for use in connection with the records for collateral loans. These forms are merely suggestive, and in this respect are like other forms presented in the various chapters on bookkeeping records. Very few banks or trust companies handle their accounts in exactly the same way, and changes and additions will therefore be necessary or advisable in making use of the forms suggested, in order to meet the particular requirements of individual companies:

Form I

LOAN TO   No.

Principal						Interest			
Date	Remarks	Vo'ch'r	Loaned	Paid Off	Balance	Rate	Accrued	Due	Collected

Form II

Collateral Security

LOAN No.

Date Received	Par	Description	Value	Date Returned

CHAPTER XVI

INTEREST ACCOUNTS

§ 189. Functions of the Three Interest Accounts

Interest is earned and accrues every day; then, at convenient periods, it *matures* and becomes collectible; then or thereafter it is collected and takes the form of cash. These three stages may be represented by the bookkeeping formulas:

(1) Interest Accrued / Interest Revenue
(2) Interest Due / Interest Accrued
(3) Cash / Interest Due

Frequently the three accounts, Interest Revenue, Interest Accrued, and Interest Due, are confused under the one title "Interest," although they have three distinct functions. Interest Revenue (which alone may be termed simply "Interest") shows how much interest has been earned during the current fiscal period. The balance of Interest Accrued shows how much of those earnings and of the earnings in previous periods has not yet fallen due. The balance of Interest Due shows how much of that which has fallen due remains uncollected.

The first of the three entries is the only one which imports a modification in the wealth of the proprietor; the other two are merely permutative, representing a shifting from one kind of asset to another. It is not the mere collecting of interest which increases wealth; nor is it merely the coming-due of the interest: it is the earning of it from day to day.

§ 190. A Double Record for Interest Earned

Interest accrued need not, and cannot conveniently, be computed on each unit of investment, as we have already stated. But it can readily be computed on all investments of the same kind and rate of interest, and the aggregate (say for a month) will form the basis for the entry "Interest Accrued / Interest Revenue." Or a daily rate for the entire investment may be established, and this may be used without change, day after day, until some change in the principal or in the rate causes a variation in the daily increment. The most complete and accurate method is to keep a double register of interest earned: first, by daily additions; second, by monthly aggregates, classified under rates and time.

§ 191. Example of Interest Income

To illustrate this, we will take a period of ten days instead of a month, and assume that the investments are in mortgages only. On the first day of the period there is \$100,000 running at 4%, \$60,000 at 4½%, and \$150,000 at 5%. On the second day, \$10,000 at 4% is paid off, and on the fifth day \$5,000 at 5%. On the seventh day a loan of \$15,000 is made at 4½%, and one of \$6,000 at 5%. We begin by computing the daily increment, as follows:

One day at 4%	on \$100,000.....	\$11.1111
One day at 4½%	on \$ 60,000.....	7.50
One day at 5%	on \$150,000.....	20.8333
	Total daily increment......	\$39.4444

§ 192. Daily Register of Interest Accruing

The decimals are carried out two places beyond the cents, and rounded only in the total. The daily register will then be conducted as follows:

DAILY REGISTER OF INTEREST ACCRUING
For the month of ...., 1914

Date	No. of Loan	Decrease in Principal	Increase in Principal	Rate	Working Column	Daily Increment
1						$ 39.4444
2					$39.4444	39.4444
"	647	$10,000		4	1.1111	
3						38.3333
4						38.3333
5					$38.3333	38.3333
"	453	5,000		5	.6944	
6						37.6388
7					$37.6388	37.6388
"	981		$ 15,000	4½	1.875	
"	982		6,000	5	.8333	
8						40.3472
9						40.3472
10						40.3472
		$15,000	$ 21,000			$390.21
			Balances at Close		Proof of Rate One day	
			$ 90,000	4		$ 10.
			75,000	4½		9.375
			151,000	5		20.9722
			$316,000			$ 40.3472

§ 193. Monthly Summary

The monthly register or summary takes up, first, the mortgages upon which payments are made, then those remaining to the end of the month, whether old or new. Its result will corroborate that of the daily register.

The monthly register or summary of interest accruing may be kept in the following form. As the loans are paid off, the interest accrued is entered up in the last column. New loans negotiated, or increases in principal, are entered in column four, and the interest accruing to date of payment is carried to the last column in a similar manner.

MONTHLY SUMMARY OF INTEREST ACCRUING
For the month of ...., 1914

Date	No. of Loan	Paid off	Remaining	Rate	Days	Monthly Increment
2	647	$10,000		4	2	$ 2.2222
5	453	5,000		5	5	3.4722
7	981		$ 15,000	4½	3	5.625
"	982		6,000	5	3	2.50
10			90,000	4	10	100.00
"			60,000	4½	10	75.00
"			145,000	5	10	201.3888
			$316,000			$390.21

§ 194. Method and Importance of Interest Earned Account

The daily and monthly registers of interest earned may be in separate books or in one book—preferably the latter in most cases. A convenient arrangement would be to use two confronting pages for a month, one and one-half pages for the daily register, and one-half page for the monthly register. If an accurate daily statement of affairs is kept, the daily interest accrued will form part of that system. Again, the interest on mortgages, on bonds, on loans, or on discounts may be separated or be all thrown together. In all such respects the individual circumstances must govern, and no precise forms can be prescribed. Our main contention is that in some manner interest should be accounted for *when earned* rather than *when collected,* or *when due.*

§ 195. Interest Accounts in General Ledger

The general ledger accounts of Interest, Interest Accrued, and Interest Due will now be illustrated in simple form as to mortgages only. It is easier to combine the several kinds of interest, when carrying them to the Profit and Loss account, than to separate them if they are all thrown in together at first.

Form I—Interest Revenue
Mortgages

1914					1914				
June	30	Carried to Profit and Loss	$4270	60	Jan.	1-31	Total Earnings	$ 654	58
					Feb.	1-28	" "	708	25
					March	1-31	" "	723	33
					April	1-30	" "	756	67
					May	1-31	" "	719	44
					June	1-30	" "	708	33
			$4270	60				$4270	60

Form II—Interest Accrued
Mortgages

1914					1914				
Jan.	0	Balance	$2362	50					
	1-31	Earnings	654	58	Jan.	1-31	Due	$1272	50
Feb.	1-28	"	708	25	Feb.	1-28	"	125	00
March	17	Cash for Accrued on No. 987	58	33					
	1-31	Earnings	723	34	March	1-31	"	875	00
April	1-30	"	756	67	April	1-30	"	625	00
May	1-31	"	719	44	May	1-31	"	1200	00
June	1-30	"	708	33	June	1-30	"	65	00
						30	Balance	2528	94
			$6691	44				$6691	44
July	0	Balance	$2528	94					

Form III—Interest Due
Mortgages

1914					1914				
Jan.	0	Balance	$ 125	00					
	1-31	Due	1272	50	Jan.	1-31	Collections	$1325	00
Feb.	1-28	"	125	00	Feb.	1-28	"	197	50
March	1-31	"	875	00	March	1-31	"	850	00
April	1-30	"	625	00	April	1-30	"	600	00
May	1-31	"	1200	00	May	1-31	"	1200	00
June	1-30	"	65	00	June	1-30	"	100	00
						30	Balance	15	00
			$4287	50				$4287	50
July	0	Balance	$ 15	00					

§ 196. Payment of Accrued Interest

There is one entry in Interest Accrued account which does not arise from earnings: the accrued interest on Mortgage No. 987, which is paid for in cash on March 17, the mortgage not having been made direct with the mortgagor, but purchased from a previous holder. This case occurs frequently in bond accounts, but not so often in connection with mortgages.

CHAPTER XVII

BONDS AND SIMILAR SECURITIES

§ 197. Investments with Fluctuating Values

The investments heretofore considered are interest bearing, but bear no premium nor discount; the variation from time to time is in the rate of interest, while the principal is invariable. When we consider investments whose price fluctuates, while the cash rate of interest is constant, the problem is more difficult, because there are several prices which it may be desired to record, viz., the original cost, the market value, the par, and the book value or amortized value. The original cost and the par are the extremes: one at the beginning, and one at the end of the investment. The book values are intermediate between these, and represent the investment value, falling or rising to par by a regular law, which maintains the net income at a constant rate. The market value is not an investment value, but a commercial one; it is the price at which the investor *could* withdraw his investment, but until he has done so, he has not profited by its rise, nor lost by its fall. So long as he retains his investment, the market value does not affect him, nor should it enter into his accounts. It is valuable information, however, from time to time, if he has the privilege of changing investments, or the necessity of realizing.

§ 198. Amortization Account

The account with principal, showing at each half-year the result of amortization, is very suitably kept in the three-

column or balance-column form recommended in § 169 for mortgages. Thus, the history of the bonds in Schedule (F), § 139, would be thus recorded in ledger form:

$100,000 SMITHTOWN 5'S OF MAY 1, 1919

Date		Dr.	Cr.	Balance
1914 May 1	Purchased from A. B. & Co.	$104,500		
Nov. 1	Amortization (4%)		$410.97	$104,089.03
1915 May 1	" "		419.19	103,669.84
Nov. 1	" "		427.57	103,242.27
1916 May 1	" "		436.12	102,806.15

§ 199. Effect on Schedule of Additional Purchases

In case of an additional purchase the account will, of course, be debited and cash credited. It will then be necessary to reconstruct the schedule from that point on. This may be done in either of two ways: (1) make an independent schedule of the new purchase, and then consolidate this with the old one, adding the terms; or (2) add together the values of the old and new bonds at the next balance date; find what the basis of the total is, eliminate any slight residue (§§ 137 to 140, inclusive), and proceed with the calculation.*

§ 200. The Bond Sales Account

In case of a sale, the procedure is different. Instead of crediting the Bond account by cash, it is best to transfer the amount sold to a Bond Sales account at its book value computed down to the day of sale; Bond Sales account will then show a debit, and the cash proceeds will be credited to the same account. The resultant will show a gain or loss on the sale, and at the balancing date the account will be closed into

---

* Bonds purchased *flat* should be separated into principal and interest.

Profit and Loss. Thus, in the example in § 198, we will suppose a sale on August 1, 1916, of half the $100,000 at 102.88, or $51,440. We find the book value of the $50,000 on August 1, which is $51,291.86; we transfer this to the debit of the Bond Sales account in the general ledger, which account we credit with the $51,440 cash proceeds. Bond Sales is purely a Profit and Loss account, and at the proper time will show the actual profit realized on the sale, $51,440 — $51,291.86 = $148.14.

FORM I—BOND LEDGER
$100,000 SMITHTOWN 5'S OF MAY 1, 1919

Date			Dr.	Cr.	Balance
1914	May 1	Purchased from A. B. & Co.	$104,500		
	Nov. 1	Amortization		$ 410.97	$104,089.03
1915	May 1	"		419.19	103,669.84
	Nov. 1	"		427.57	103,242.27
1916	May 1	"		436.12	102,806.15
	Aug. 1	Sale to C. D. & Co.			
		$50,000 @ 102.88		51,291.86	51,514.29
	"	Amortization on $50,000		111.21	51,403.08
	Nov. 1	" on balance		222.43	51,180.65

FORM II—GENERAL LEDGER
BOND SALES

1914 Aug. 1	Smithtown 5's	$51,291.86	1914 Aug 1	Proceeds	$51,440.00

To adjust the profit in the Bond account itself would be as unphilosophical as the old-fashioned Merchandise account before the Purchases and Sales accounts were introduced, and even more awkward.

§ 201. Requirements as to Bond Records

Besides the book value of a bond, the par is also needed because the cash interest is reckoned upon the par. For some purposes, also, it is useful to show the original cost. We must, therefore, provide means for exhibiting these three values: the par, the original cost, and the book value. A mere memorandum of par and cost at the top would be sufficient where the group of bonds in question will all be held to the same date; but this is not always the case, and provision must be made for increase and decrease. The three-column form of ledger (§ 169), constantly exhibiting the balance, is the most suitable for this purpose also. But if we endeavor to display all of these forms side by side, we require nine columns, and this makes an unwieldy book. The most practical way is to abandon the use of debit and credit columns, and proceed by addition and subtraction, or in what the Italians term the *scalar* (ladder-like) form, which gives a perfectly clear result, especially if the balances are all written in red. Headed by a description of the bonds, and embracing, also, a place for noting the market value at intervals (not as matter of account, but of information), the Principal account will appear as shown in Form I (page 152).

§ 202. Form of Bond Ledger

As far as the bond ledger is concerned, the transfer of the $50,000 sold to Sales account is final; we have, however, in the example indicated (§ 200), a way of incorporating a statement of the profit or loss in the margin for historical purposes. The amortization of November 1, 1916, is composed of two parts: 3 months on the $50,000 sold, $111.21; and the regular 6 months on the $50,000 retained, $222.43. In the example given in § 200, these are entered separately; either method may be pursued, but on the whole there are greater advantages in postponing all entries of amortization

FORM I—PRINCIPAL ACCOUNT

Obligation of And issued by } *The Village of Smithtown, Ohio*

Name of obligation, *Sewer Bonds* — Date, *May 1, 1914*
Where payable, *at the First National Bank, New York*
When payable, *May 1, 1919*
Interest at *5%* per annum, payable each *1st of May and November*
How payable, *by Coupon* — Net income, *4%*
Numbers of Bonds, *51 to 150, inclusive*

Date	Voucher	Principal	Par Value	Original Cost	Book Value	Market Value
1914 May 1	2572	Bought from A. B. & Co.	$100,000.00	$104,500.00	$104,500.00	
Nov. 1		Amortization			410.97	
					$104,089.03	$104,000.00
1915 May 1		"			419.19	
					$103,669.84	103,875.00
Nov. 1		"			427.57	
					$103,242.27	103,000.00
1916 May 1		"			436.12	
					$102,806.15	103,000.00
Aug. 1	J 697	Sale to C. D. & Co. for $51,440.00	50,000.00	52,250.00	51,291.86	
		Profit 148.14	$ 50,000.00	$ 52,250.00	$ 51,514.29	
Nov. 1		Amortization			333.64	
					$ 51,180.65	51,200.00

till the end of the half-year. The three months' amortization of the bonds sold is in effect implied in the price $51,-291.86, which is reduced by the amortization ($111.21) from $51,403.07, the half of $102,806.15, but it need not be entered till November 1.

§ 203. Interest Due Account

The register of interest due on bonds is conducted on precisely the same principles as that described for mortgages in § 183; in fact, they are but subdivisions of the same register. Of course, only the cash interest is considered.

§ 204. Interest Account—Bond Ledger

The interest pages of the bond ledger are also similar to those of the mortgage ledger (§ 183), but the dates of interest due may be printed in advance, there being but little chance of partial payments disturbing their orderly arrangement.

The paging of the bond ledger will probably be geographical, as far as possible, in respect of public issues, and alphabetical in respect of those of private corporations. The loose-leaf plan permits an indefinite number of classifications from which to choose. The date tags suggested in § 178 are especially useful for pointing out dates for interest falling due, as "J J," "F A," etc.

§ 205. Amortization Entries

The entries of amortization are made directly from the schedules of amortization, the preparation of which was discussed in Chapter X. But it is necessary, also, to make up a list of these several amortizations in order to form the general ledger entry:

Amortization / Bonds
or, Amortization / Premiums

according to the form of the general ledger. This list should be in the same order as the bond ledger. Probably the most practical way is to combine it with the trial balance of the bond ledger, thus giving at each fiscal period a complete list of the holdings, which may give the par, cost, book, and market values, the titles of the securities being written but once. The total of the second column will form the basis of the entry for amortization. The next three columns will corroborate the general ledger balances.

Bond Statement for the Half-Year Ending.........

Name and Description	Amortization	Book Value	Par Value	Original Cost	Market Value

In the cases of bonds bought at a discount, the analogous general ledger entry would be:

Bonds / Accumulation
or, Discounts / Accumulation

We have provided in the preceding form for amortization only and not for accumulation on bonds below par. Where the latter values are few in number they may be embraced in the same column, but distinguished as negatives by being written in red or encircled. If the bonds below par are numerous, there should be two columns: "amortization" and "accumulation."

§ 206. Bond Entries in General Ledger

While the book value is the proper one to be introduced

into the general ledger, the par is very necessary, and sometimes the cost, and these requirements inevitably introduce some complexity. There are two methods effecting the purpose:

(1) By considering the par and cost as extraneous information, and ruling side columns for them beside the book value.

(2) By dividing the account into several accounts, by the proper combination of which the several values may be obtained.

The first plan will preserve the conformity of the Bonds account with the bond ledger better than the other. The Bonds account may, if necessary, be extended across both pages of the ledger, to allow for three debit and three credit columns, if all are required.

The second method will commend itself more to those having a repugnance to introducing into the general ledger any figures beyond those actually forming part of the trial balance. The theory on which it is based is that the premium is not part of the bond, but is a sum paid in advance for excess interest, while the discount is a rebate returned to make good deficient interest. This is a perfectly admissible way of looking at the matter, especially from the personalistic point of view; for the debtor does not owe us the premium and has nothing to do with it. Still the other view, which regards the investment as a whole, is also correct, and we may adopt whichever is most suitable to our purpose.

§ 207. Accounts Where Original Cost Is Disregarded

If original cost is disregarded, or deemed easily obtainable when required, the accounts may be:

(a) Bonds at Par
(b) Premiums
(c) Discounts

or,

(a) Bonds at Par
(b) Premiums and Discounts

If premiums and discounts are kept separate, Premiums account must always show a debit balance, being credited for amortization; Discounts account must show a credit balance, being debited for accumulation. If the two are consolidated, only the net amortization will be credited (§ 205); or if the greater part of the bonds were below par, the net accumulation only would be debited. The choice between one account and two for premiums and discounts is largely a question of convenience.

The management of such a double or triple account is obvious, entries of transactions being divided between par and premiums, or par and discounts, but we give in § 214 an example of each.

We shall hereafter confine the discussion to premiums, leaving the cases of discount to be determined by analogy.

§ 208. Amortization Reserve

Where it is deemed necessary to keep account of cost also, as well as of par and book value, the difficulty is somewhat greater, as we have a valueless or extinct quantity to record, namely, so much of the original premium on bonds still held as has not yet been absorbed in the process of amortization. This carrying of a dead value, which is somewhat artificial, necessitates the carrying, also, of an artificial annulling or offsetting account, the sole function of which is to express this departed value. We may call this credit

account "Reserve for Amortization." It is analogous to Depreciation and Reserve for Depreciation. The part of the premiums which has been extinguished by credits to Reserve for Amortization may be designated as "Premiums Amortized," or "Ineffective Premiums," while the live premiums may be styled "Effective Premiums," being what in § 207 we called simply "Premiums." A double operation takes place in these accounts: first, the absorption of effective premiums by lapse of time; and second, the cancellation of ineffective premiums upon redemption or sale.

§ 209. Premiums and Amortization

There are two ways of handling these accounts, differing as to premiums. We may keep two accounts: "Effective Premiums" and "Amortized Premiums," or we may combine these in one, "Premiums at Cost." The entire scheme will be:

(a) Bonds at Par
(b) Premiums at Cost
        (e) Reserve for Amortization

or,

(a) Bonds at Par
(c) Effective Premiums
(d) Amortized Premiums
        (e) Reserve for Amortization

"a" will in both schemes be the same; "e" will also be the same. "b" is the sum of "c" and "d." In the former, the cost is $a + b$, while the book value is $a + b - e$. In the latter the book value is $a + c$, while the cost is $a + c + d$. The former gives the cost more readily than the latter, and the book value less readily. The former might be considered the more suitable for a trustee; the latter, for an investor.

Account (a), Bonds at Par, is debited for par value of purchases and credited for par value of sales. Its only two entries are:

Bonds at Par / Cash (or some other asset)
Cash (or some other asset) / Bonds at Par

In case of purchase at a premium, the premium is charged to Premiums at Cost or to Effective Premiums, as the case may be, there being no ineffective premiums at this time.

§ 210. Writing Off Premiums

When premiums are written off, on the first plan illustrated in § 209 there is but one entry: crediting Reserve for Amortization and debiting the Profit and Loss account or its subdivision.

Amortization / Reserve for Amortization

The second plan involves not only this process, but a transfer from Effective to Amortized Premiums. Thus, the aggregate of premiums written off is posted four times as a consequence of the separation of premiums at cost into two accounts:

Premiums Amortized / Effective Premiums
Amortization / Reserve for Amortization

§ 211. Disposal of Amortization

The word "Amortization" has been used in the illustrative entries as the title of an account tributary to Profit and Loss. At the balancing period it may be disposed of in either of two ways: It may be closed into Profit and Loss direct; or it may be closed into Interest account, the balance of which will enter into Profit and Loss at so much les-

sened a figure. By the former method the Profit and Loss account will show, on the credit side, the gross cash interest, and on the debit side the amount devoted to amortization; the second method exhibits only the net income from interest on bonds. Whether it be preferable to show both elements, or only the net resultant, will be determined by expediency.

§ 212. Amortization Accounting—Comparison of Methods

In §§ 200 and 202 we discussed two methods of keeping account of amortization: the first (in § 200), where any incidental amortization occurring in the midst of the period is at once entered; the second (in § 202), where all such entries are deferred to the end of the period, and comprised in one entry in the general ledger. If the latter method be adopted, the Amortization account may be dispensed with altogether, and the total amount amortized (which is credited to Bonds, or to Premiums, or to Reserve for Amortization) may be debited at once to Profit and Loss or to Interest, without resting in a special account. A single item, of course, needs no machinery for grouping.

§ 213. Irredeemable Bonds a Perpetual Annuity

Irredeemable bonds (§ 146) merely lack the element of amortization, and require no special arrangement of accounts. The par is purely ideal, as it never can be demanded and is merely a basis for expressing the interest paid. What the investor buys is a perpetual annuity. If he buys an annuity of $6 per annum, it is unimportant whether it is called 6% on $100 principal, or 4% on $150 principal; and this $150 may be the par value, or it may be $100 par at 50% premium, or $200 par at 25% discount. The par value is really non-existent.

§ 214. Bond Accounts for General Ledger

In the present section are shown the forms for the general ledger outlined in §§ 206-212. We will suppose that on January 1, 1915, the following lots of bonds are held:

JANUARY 1, 1915

Par		Book Value
$100,000	5% Bonds, J J, due Jan. 1, 1925, net 2.7%; value..	$120.039.00
	Original cost, $124,263.25	
100,000	3% Bonds, M N, due May 1, 1918, net 4%; value..	96,909.10
	Original cost, $93,644.28	
10,000	4% Bonds, A O, due Oct. 1, 1916, net 3%; value..	10,169.19
	Original cost, $10,250.00	
$210,000	Totals	$227,117.29

The premiums on the 5% and 4% bonds amount to $20,208.19. The discount on the 3% bonds is $3,090.90. The net premium is $17,117.29. The total original cost was $228,157.53.

Bond Accounts for General Ledger—Plan I (§ 206)

Dr. Bonds Account *Cr.*

		Par	Cost	Book Value			Par	Cost	Book Value
1915					1915				
Jan. 0,	Balances	$210,000.00	$228,157.53	$227,117.29	June 30,	Amortization			$ 488.76
					Dec. 31,	"			492.59
					1916				
					June 30,	"			496.40
					Oct. 1,	Redeemed	$ 10,000,00	$ 10,250.00	10,000.00
					Dec. 31,	Amortization			475.21
					1917				
					June 30,	"			453.63
					Dec. 31,	"			456.68
					"	Balances......	200,000.00	217,907.53	214,254.02
		$210,000.00	$228,157.53	$227,117.29			$210,000.00	$228,157.53	$214,254.02
1918									
Jan. 0,	Balances	$200,000.00	$217,907.53	$214,254.02					

BOND ACCOUNTS FOR GENERAL LEDGER—PLAN II (§ 207)

Dr. BONDS AT PAR *Cr.*

1915		1916	
Jan. 0,	Balance.......$210,000.00	Oct. 1,	Redeemed......$10,000.00

Dr. PREMIUMS *Cr.*

1915		1915	
Jan. 0,	Balance........$20,208.19	June 30,	Amortization....$926.94
		Dec. 31,	" 939.54
		1916	
		June 30,	" 952.28
		Dec. 31,	" 940.21
		1917	
		June 30,	" 927.93
		Dec. 31,	" 940.47

Dr. DISCOUNTS *Cr.*

1915		1915	
June 30,	Accumulation....$438.18	Jan. 0,	Balance.........$3,090.90
Dec. 31,	" 446.95		
1916			
June 30,	" 455.88		
Dec. 31,	" 465.00		
1917			
June 30,	" 474.30		
Dec. 31,	" 483.79		

Bond Accounts for General Ledger—Plan III (§ 207)
(Original cost omitted)

Dr. Bonds at Par *Cr.*

1915		1916	
Jan. 0,	Balance.......$210,000.00	Oct. 1,	Redeemed......$10,000.00

Dr. Premiums and Discounts *Cr.*

1915		1915	
Jan. 0,	Balance........$17,117.29	June 30,	Amortization....$488.76
		Dec. 31,	" 492.59
		1916	
		June 30,	" 496.40
		Dec. 31,	" 475.21
		1917	
		June 30,	" 453.63
		Dec. 31,	" 456.68

BOND ACCOUNTS FOR GENERAL LEDGER —PLAN IV (§ 209)
("Bonds at Par" as in foregoing plans)

Dr. PREMIUMS AT COST *Cr.*

1915			1916		
Jan. 0,	Balance........	$18,157.53	Oct. 1,	Canceled at Redemption....$	250.00
			1917		
			Dec. 31,	Balance.......	17,907.53
		$18,157.53			$18,157.53
1918					
Jan. 0,	Balance........	$17,907.53			

Dr. RESERVE FOR AMORTIZATION *Cr.*

1916			1915		
Oct. 1,	Canceled at Redemption....$	250.00	Jan. 0,	Balance........	$1,040.24
			June 30,	Amortization...	488.76
			Dec. 31,	" ...	492.59
1917					
Dec. 31,	Balance.......	3,653.51	1916		
			June 30,	" ...	496.40
			Dec. 31,	" ...	475.21
			1917		
			June 30,	" ...	453.63
			Dec. 31,	" ...	456.68
		$3,903.51			$3,903.51
			1918		
			Jan. 0,	Balance.......	$3,653.51

BOND ACCOUNTS FOR GENERAL LEDGER—PLAN V (§ 210)
(By the balance column method)

	BONDS AT PAR	*Dr.*	*Cr.*	*Balance Dr.*
1915				
Jan. 0	Balance................	$210,000.00		$210,000.00
1916				
Oct. 1	Redemption............		$10,000.00	200,000.00

	EFFECTIVE PREMIUMS	*Dr.*	*Cr.*	*Balance Dr.*
1915				
Jan. 0	Balance................	$ 17,117.29		$ 17,117.29
June 30	Amortized.............		$ 488.76	16,628.53
Dec. 31	"		492.59	16,135.94
1916				
June 30	"		496.40	15,639.54
Dec. 31	"		475.21	15,164.33
1917				
June 30	"		453.63	14,710.70
Dec. 31	"		456.68	14,254.02

	INEFFECTIVE OR AMORTIZED PREMIUMS	*Dr.*	*Cr.*	*Balance Dr.*
1915				
Jan. 0	Balance................	$ 1,040.24		$ 1,040.24
June 30	Amortized.............	488.76		1,529.00
Dec. 31	"	492.59		2,021.59
1916				
June 30	"	496.40		2,517.99
Oct. 1	Canceled by Redemption		$ 250.00	2,267.99
Dec. 31	Amortized.............	475.21		2,743.20
1917				
June 30	"	453.63		3,196.83
Dec. 31	"	456.68		3,653.51

Reserve for Amortization		Dr.	Cr.	Balance Cr.
1915				
Jan. 0	Balance................		$ 1,040.24	$ 1,040.24
June 30	Amortized.............		488.76	1,529.00
Dec. 31	"		492.59	2,021.59
1916				
June 30	"		496.40	2,517.99
Oct. 1	Canceled by Redemption	$ 250.00		2,267.99
Dec. 31	Amortized.............		475.21	2,743.20
1917				
June 30	"		453.63	3,196.83
Dec. 31	"		456.68	3,653.51

CHAPTER XVIII

DISCOUNTED VALUES

§ 215. Securities Payable at Fixed Dates Without Interest

The securities heretofore considered have all carried a stipulated rate of interest or annuity. There is another class to which no periodical interest attaches, but the obligation is simply to pay a single definite sum on a certain date. The present value of that sum at the current or contractual rate of income is, of course, obtained by discounting according to the principles explained in Chapter II. If the maturity were more than one year distant at the time of discount, it would be necessary to compute the compound discount; but in practice this never occurs, such discounts being for a few months.

The obligations discounted in this manner are almost invariably promissory notes. Formerly they consisted largely of bills of exchange; hence the survival in bookkeeping of the words "Bills Receivable," "Bills Payable," and "Bills Discounted."

These obligations belong rather to mercantile and banking accountancy than to investment accountancy. The arrangement of accounts for recording their amounts, classification, and maturity has been so fully treated in works on those branches that we refer to them here only for the purpose of illustrating another phase of the process of securing income.

§ 216. Rates of Interest and Discount

The difference between the rate of interest and the rate

of discount has been pointed out in Chapter II. It was there shown that in a single period the rate of interest 3% corresponds to the rate of discount .029126. Hence, if we discount a note for $1.00 at 2.9126%, we acquire interest at the rate of 3% on the $.970874 actually invested. The rate of interest is always greater than the rate of discount.

§ 217. Rate of Discount Named in Notes

It is usual to name a rate of discount rather than a rate of interest in stipulating for the acquisition of notes. For example, a three months' note for $1,000 is taken for discount at 6% (per annum). This means that $.015 is to be retained by the payee of the note from each dollar, and the amount actually paid over is $985. The income from this is the $15, and by dividing 15 by 985 we readily ascertain that the rate of interest realized is 6.09%. It is sometimes believed that there is a kind of deception in this; that the borrower agrees to pay 6% and actually has to pay 6.09%. But this is not so: the bargain is not to pay 6% interest, but to allow 6% discount, which is a different thing.

§ 218. Form as Affecting Legality

Curiously, the lawfulness or unlawfulness of a transaction sometimes depends upon the mere form of words in which it is expressed. Thus, suppose that A lends $985 to B, who promises to repay $1,000 at the end of 3 months. If B's promise reads: "I promise to pay $1,000," A is a law-abiding citizen; but if B writes: "I promise to pay $985 and interest at 6.09% per annum," the statute prohibiting usury is violated.

§ 219. Entry of Notes Discounted

Notes discounted are usually entered among the assets at the full face, and the discount credited to an offsetting

account, "Discounts," the latter having precisely the same effect as the Discounts account used in connection with bonds. The difference of the two is the net amount of the asset. Strictly speaking, the discount is at first an offset to the note, and represents at that time nothing earned whatever; as time goes on, the earning is effected by diminution of this offset, which is equivalent to a rise in the net value of the note, from cost to par. In § 220 the process is shown by the state of the accounts at the initial date and at the end of each month up to maturity, for a 3 months' note for $1,000, discounted at 6%.

§ 220. Discount and Interest Entries

(1) When Discounted

Note		Discount	
$1000.00			$15.00

(2) At the End of One Month

Note		Discount		Interest Revenue	
$1000.00		$ 5.00	$15.00		$5.00

(3) At the End of Two Months

Note		Discount		Interest Revenue	
$1000.00		$ 5.00	$15.00		$5.00
		5.00			5.00

(4) At Maturity

Note		Discount		Interest Revenue	
$1000.00		$ 5.00	$15.00		$5.00
		5.00			5.00
		5.00			5.00
		$15.00	$15.00		

§ 221. Total Earnings from Discounts

Since notes are issued generally for short periods, the gradual crediting of earnings illustrated in § 220 is usually ignored. At the date when the books are closed, an inventory should be taken of the discounts unearned. The difference between the amount of this inventory and the net credit in the Discounts account represents the earning from discounts during the fiscal period, and this earning should then be transferred to Profit and Loss. The unearned discounts may be easily computed by finding the discount on each note from the date of closing the books to the respective dates of maturity. The investment value of the notes on hand at the close of the fiscal period will be the difference between the par and the unearned discount. Expressed in a formula, the earnings from discounts may be found as follows:

Unearned discounts at beginning of fiscal period,
$+$ discounts credited during period,
$-$ unearned discounts at end of period,
$=$ earnings from discounts during period.

Part II—Problems and Studies

CHAPTER XIX

INTEREST AND DISCOUNT

§ 222. **Problems in Simple Interest***

(1) What is the time in months and days from January 10th to:

- (a) June 12th?
- (b) July 4th?
- (c) September 1st?

(2) What date is:

- (a) Two months after June 30th?
- (b) Four months after May 31st?
- (c) Two months after December 31st?
- (d) Five months and seven days after September 26th?

(3) On a loan of $54,750, interest payable semi-annually at 4% per annum, interest was last paid to and including November 1: compute the interest accrued on the following February 25th:

- (a) In the customary manner, legal in New York before 1892.
- (b) Assuming that the odd days are 365ths of a year.

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* In connection with the text of Chapter II. For answers see § 224.

(c) Compute the same by both methods at 4½%.
(d) " " " " " " " 5%.
(e) " " " " " " " 6%.

(4) On a 365-day basis, the interest for 17 days, on a certain sum, at a certain rate, was $83.73; what would have been the interest on a 360-day basis?

(5) The interest for 19 days on a certain sum at a certain rate was $2,185.00 on a 360-day basis; compute the interest on a 365-day basis.

§ 223. Notes on the One Per Cent Method

Observe that when days are considered as 360ths of a year, it is useful to know how many days correspond to one per cent. For example, if the rate is 3%, it takes 120 interest days to earn 1% interest.

At 3 %	the	number	of	days	for	1%	is	120
At 4 %	"	"	"	"	"	"	"	90
At 4½%	"	"	"	"	"	"	"	80
At 5 %	"	"	"	"	"	"	"	72
At 6 %	"	"	"	"	"	"	"	60
At 8 %	"	"	"	"	"	"	"	45
At 9 %	"	"	"	"	"	"	"	40

For purposes of calculation we may set down the number of days corresponding to 1% at the given rate, and in line with it the principal, pointing off two places from the right in the principal in order to obtain 1%. Thus, in Problem (3) of the preceding section:

90 days $547.50

meaning that the interest for 90 days at 4% is $547.50. Knowing the interest for 90 days, we can build up that for 114 days (3 months and 24 days on the 360-day basis). 24 days = 15 days + 9 days. 15 days is 1/6 of 90 days; 9

days, 1/10. Dividing the interest for 90 days by 6 to secure the interest for 15 days, and by 10 to secure the interest for 9 days, and adding, gives the result:

90	days	$547.50
15	"	91.25
9	"	54.75
114	"	$693.50

The same result may be obtained, and just as easily, by the combination 90 + 18 + 6. Sometimes the work may be shortened by the use of subtraction; in the present case, no time would be saved by this method, the result working out as follows:

	90 days			$547.50
	30 "		$182.50	
less	6	(1/5 of 30)	36.50	146.00
	114			$693.50

In the case of problem (3-d), on the 5% basis, the result would work out as follows:

72	days		$547.50
18	"	(¼ of 72)	136.875
24	"	(⅓ of 72)	182.50
114	"		$866.875

Rates like 7% or 3½%, which are not exact divisors of 360, must be obtained from the exact rates by division and addition. Thus, 7% is derived by adding 1/6 to 6%, which is obtained as follows:

60	days	$547.50	
30	"	273.75	
20	"	182.50	
4	"	36.50	
114	"	$1,040.25	(interest at 6%)
add 1/6		173.375	
		$1,213.625	(interest at 7%)

§ 224. Answers to Problems in Simple Interest

Problem (1)

(a) 5 months, 2 days
(b) 5 months, 24 days
(c) 7 months, 22 days

Problem (2)

(a) August 30th
(b) September 30th
(c) February 28th or 29th
(d) March 4th or 5th

Problem (3)

(a) $693.50
(b) $691.50
(c) $780.19 (360-day method)
$777.94 (365-day method)
(d) $866.875 (360-day method)
$864.375 (365-day method)
(e) $1,040.25 (360-day method)
$1,037.25 (365-day method)

Problem (4)

$84.89

Problem (5)

$2,155.07

§ 225. Problems in Compound Interest*

(6) Find the amount of $1 at 2% per period, correct to six decimals:

(a) For one period
(b) For two periods
(c) For three periods
(d) For four periods
(e) For five periods

(7) Find the present worth of $1 at 2%, correct to six decimals:

(a) For one period
(b) For two periods
(c) For three periods
(d) For four periods
(e) For five periods

In § 29, several methods are mentioned for finding the present worth; assuming that the solutions for problem (6) above have been found, the easiest method of finding the present worth for five periods would be to divide 1 by the amount for five periods. The present worths for 1, 2, 3, and 4 periods can then be found by multiplying the present worth for five periods successively by 1.03. This is much easier than dividing 1 successively by 1.03. 1.02

(8) Find the amount of $1 at 1¾% (.0175) per period:

(a) For one period
(b) For two periods
(c) For three periods
(d) For four periods
(e) For five periods
(f) For six periods

* In connection with the text of Chapter II. For answers see § 226.

(9) Find the present worth of $1 at 1.75% per period:

(a) For one period
(b) For two periods
(c) For three periods
(d) For four periods
(e) For five periods
(f) For six periods

(10) Find the amount and the present worth of $1,000.00 for eight periods at 1.5% per period.

(11) What is the rate of discount corresponding to 2% interest?

(12) What is the rate of interest corresponding to the discount rate of .0384615?

(13) Three notes for $1,000.00 each, due (without interest) at three months, six months, and one year respectively, are discounted at 6%:

(a) If the proceeds of the first note are $985, find the equivalent interest rate.
(b) If the proceeds of the second note are $970, find the equivalent interest rate.
(c) If the proceeds of the third note are $940, find the equivalent interest rate.

(14) What is the compound interest on $1 for five periods at 2%?

(15) What is the compound discount on $1 for four periods at 2%?

§ 226. Answers to Problems in Compound Interest

Problem (6)

(a) $1.02
(b) $1.0404
(c) $1.061208
(d) $1.082432
(e) $1.104081

Problem (7)

(a) \$.980392
(b) \$.961169
(c) \$.942322
(d) \$.923845
(e) \$.905731

Problem (8)

(f) \$1.109702

Problem (9)

(f) \$.901143

Problem (10)

Amount, \$1,126.493; present worth, \$887.711

Problem (11)

.0196078 (Observe that this decimal when divided by 2%, the rate of interest, gives the present worth for one period, .98039. This will be a test for all similar computations.)

Problem (12)

4%

Problem (13)

(a) 1.52284% quarterly, or (nominally) 6.09137% annually
(b) 3.09278% semi-annually, or (nominally) 6.18557% annually
(c) 6.38298% annually

Problem (14)

\$.104081

Problem (15)

\$.076155

§ 227. Proof of Amount and Present Worth

The amount and the present worth of the same sum for

the same time and rate should, when multiplied together, give the product 1.

Problems (6) and (7) give the amount of $1 for 5 periods at 2% as $1.104081, and its present value for the same time and rate as $.905731. These numbers multiplied together should give as a product, unity. Such multiplications of decimal numbers are best performed by beginning at the *left* of the multiplier.

1.10408	1
.905731	
993672	9
5520	405
772	8567
33	12243
1	104081
1.000000	388211

The vertical line is drawn to cut off the figures beyond the 6th decimal, which have no utility except to furnish a carrying amount for the 6th figure. They may be dispensed with by using contracted multiplication.

§ 228. Contracted Multiplication

In this process the subproducts are shortened at each step by one figure, taking into account, however, the carrying amount from the rejected figures.

	1.104081
	.905731
(first 6 figures × 9)....	993673
(first 5 figures × 0)....	0
(first 4 figures × 5)....	5520
(first 3 figures × 7)....	773
(first 2 figures × 3)....	33
(first figure × 1)....	1
	1.000000

Here we commence to multiply by 9 at the sixth figure, 8; the product would be 72, but we know that the rejected 1, $\times$ 9, would make the product nearer 73; this subproduct, therefore, becomes 993673. In each of these partial products the last retained figure is slightly increased, if necessary, by mental allowance for the next rejected figure. The last figure of the final product will, even then, not always be exact, but may vary one or two units from the correct product. In all multiplications by rounded decimals, there is an error, small it is true, in the product; this final error may be reduced to as small a quantity as desired, by increasing the decimal places in the factors to such extent as the accuracy of the work may require.

It sometimes happens in contracted multiplication that you "lose your place" and forget at what figure of the multiplicand to begin next. This may be overcome by ticking off each figure as you have done with it; or by repeating the multiplier figures from left to right and (at the same time) the multiplicand figures from right to left. In the above illustration the correlated figures would be 9-8, 0-0, 5-4, 7-0, 3-1, and 1-1.

§ 229. Problems in Use of Logarithms*

The following problems are elementary and the 4-place table given in § 43 may be used in their solution.

(16) What is the logarithm of:

(a) 3	(d) 1.8	(g) .54
(b) 30	(e) .0018	(h) 1.03
(c) 3,000	(f) 5.4	

(17) Give the number whose logarithm is:

(a) .1614	(c) 1.6474	(e) $\bar{3}.6474$
(b) 2.3838	(d) $\bar{1}.6474$	(f) .0212

* In connection with the text of Chapter III. For answers see § 231.

(18) Find the logarithm of:
(a) 291.5 (b) 4.362 (c) .027433

(19) Find the number whose logarithm is:
(a) 2.5849 (b) $\bar{1}.38425$ (c) 3.6931

(20) Prove by logarithms that:
(a) $9 \times 8 = 72$
(b) $7 \times 1.12 = 7.84$
(c) $.032 \times 300 = 9.6$
(d) $.004 \times 4000 = 16$

(21) Show by logarithms that:
(a) $72 \div 2.4 = 30$
(b) $12.5 \div 625 = .02$
(c) $5.2 \div .04 = 130$

(22) What is the 28th power of:
(a) 1.02 (b) 1.04

(23) What is the present worth of $1 for 45 periods at:
(a) 3% (b) 5%

(24) Find by logarithms the value of the following:
$829 \times 76.3 \times .0484 \div 7.28 \div 25$

§ 230. Problems Requiring Use of More Extended Tables of Logarithms*

For further exercise in logarithmic computations, Problems (14) to (18) inclusive should again be worked out, using logarithms to the limit of such tables as may be at hand. The logarithms of all of the ordinary ratios of increase $(1 + i)$, with which the operation always begins, will be found in Part III. These logarithms have been computed to fifteen places of decimals.

The following examples, which are for too many periods to be worked out arithmetically, may also be worked by

* For answers see § 231.

logarithms. If no other tables are available, the four-place tables in § 43 may be used, although these tables cannot be relied upon to bring correct results to as many decimal places as are given in the solutions.

(25) Find the amount and present worth of $1:

(a) At 1.25% for 30 periods
(b) At 1.70% for 50 periods
(c) At 2.00% for 10 periods
(d) At 2.40% for 68 periods
(e) At 2.50% for 70 periods

§ 231. Answers to Problems in Logarithms

Problem (16)

(a) .4771	(d) .2553	(g) $\bar{1}.7324$
(b) 1.4771	(e) $\bar{3}.2553$	(h) .0128
(c) 3.4771	(f) .7324	

Problem (17)

(a) 1.45	(c) 44.4	(e) .00444
(b) 242	(d) .444	(f) 1.05

Problem (18)

(a) 2.4647 (b) .6397 (c) $\bar{2}.4383$

Problem (19)

(a) 384.5 (b) .24225 (c) 4933

Problem (20)

(a) log. 9 = .9542; log. 8 = .9031. The sum of these two logarithms is 1.8573, which is the logarithm of 72. Similarly for (b), (c), and (d).

Problem (21)

(a) log. 72 = 1.8573; log. 2.4 = .3802. The first logarithm minus the second is 1.4771, which is the logarithm of 30. Similarly for (b) and (c).

Problem (22)

(a) log. 1.02 is .0086; multiplied by 28 gives .2408, which logarithm corresponds to the number 1.741. The correct result to eight decimal places is given in Part IV, being 1.74102421.

(b) 2.99; to eight decimal places, the result is 2.99870332.

Problem (23)

(a) log. 1.03 is .0128, which multiplied by 45 gives .5760. Then log. $(1 \div 1.03^{45}) =$ zero minus .5760, or $\bar{1}.4240$. The number corresponding to this last logarithm is .265; to eight places the result is .26443862. ?

(b) .111, and to eight places, .11129651.

Problem (24)

	log. 829	=	2.9186
plus	log. 76.3	=	1.8825
plus	log. .0484	=	$\bar{2}$.6848
minus	log. 7.28	=	.8621
minus	log. 25	=	1.3979
	Net result	=	1.2259,

which is the logarithm corresponding to the number 16.8; the result by actual multiplication and division is 16.82105.

Problem (25)

(a)	Amount,	$1.45161336;	present	worth,	$.68888867
(b)	"	$2.32299164;	"	"	$.43047938
(c)	"	$1.21899442;	"	"	$.82034830
(d)	"	$5.01645651;	"	"	$.19934390
(e)	"	$5.63210286;	"	"	$.17755358

CHAPTER XX

PROBLEMS IN ANNUITIES AND IN NOMINAL AND EFFECTIVE RATES

§ 232. Problems in Annuities*

(26) Find the amounts and present worths of an annuity of $1:

(a) At 1.25% for 30 periods
(b) At 1.70% for 50 periods
(c) At 2.00% for 10 periods
(d) At 2.40% for 68 periods
(e) At 2.50% for 70 periods

In Problem (26), a to e inclusive, assume that the present worth in each case is a loan, and construct a schedule showing the gradual repayment of this loan at $1 per period, for a few periods or for the entire time.

§ 233. Answers to Problems in Annuities

Problem (26)

(a)	Amount,	$36.129069;	present	worth,	$24.888906
(b)	"	$77.823037;	"	"	$33.501213
(c)	"	$10.949721;	"	"	$ 8.982585
(d)	"	$167.352355;	"	"	$33.360671
(e)	"	$185.284114;	"	"	$32.897857

* In connection with the text of Chapters IV and V.

§ 234. Problems in Rent of Annuity and Sinking Fund*

(27) What is the rent of an annuity of 30 periods valued at $1,000 if the rate of interest is 1.25% per period? In other words, what is each term of an annuity the present worth of which is $1,000, the interest earned being 1.25% per period and the number of periods 30?

(28) Assume the same present worth as in (27), and find the rent of an annuity under the following conditions:

(a) 1.70%, 50 periods
(b) 2.00%, 10 periods
(c) 2.40%, 68 periods
(d) 2.50%, 70 periods

(29) What is the sinking fund to be reserved at the end of each period and invested at 1.25%, to amount to $1,000 at the end of 30 periods?

(30) Compute the sinking funds for the same data as in (a), (b), (c), and (d), in (28) above.

(31) What amount should be laid aside each half-year to amount to $100,000 at the end of 50 years at 4% per annum, interest payable semi-annually?

(32) What amount at 3%?

(33) A father wishing to make a gift of $10,000 to his son, now 15 years old, on the latter's 21st birthday, deposits a certain sum at a trust company, on a 4% annual basis, on the 16th and each succeeding birthday, including the 21st, sufficient to amount to the $10,000 when the last deposit is made. Find the required annual deposit.

(34) Assume that after the annual deposit is made on the 18th birthday, the trust company states that the interest rate thereafter on deposits is to be only 3% annually. Find the annual amount which should be deposited on the 19th,

* In connection with the text of Chapter VII.

20th, and 21st birthdays in order to reach the desired \$10,000.

(35) On July 1, 1914, a company decides to accumulate a sinking fund of \$100,000 by July 1, 1921, assuming that interest on the fund will be at the rate of 4% per annum. It is expected that annual contributions to the fund of \$12,000 each will be made at July 1, 1917, 1918, 1919, 1920, and 1921. Find the two equal contributions required at July 1, 1915 and 1916, in order that the seven contributions, with accumulated interest, may amount to \$100,000 at July 1, 1921.

§ 235. Answers to Problems in Rent of Annuity and Sinking Fund

Problem (27)
\$40.17854

Problem (28)
(a) \$29.84967
(b) \$111.32653
(c) \$29.975416
(d) \$30.39712

Problem (29)
\$27.67854

Problem (30)
(a) \$12.84967
(b) \$91.32653
(c) \$5.975416
(d) \$5.39712

Problem (31)
\$320.27

Problem (32)
\$437.06

Compare the answers to Problems (27) and (29); (28-a) and (30-a); (28-b) and (30-b); (28-c) and (30-c); and (28-d) and (30-d), respectively. Note that the differences between these five pairs of answers are in proportion to the respective five rates of income.

Problem (33)

$1,507.62

Problem (34)

$1,571.53

Problem (35)

$14,103.35

§ 236. Problems in Nominal and Effective Rates*

(36) If the interest rate is 12% per annum, payable in monthly instalments, what is the effective annual rate?

(37) If the interest is 12% payable semi-annually, what is the effective annual rate?

(38) What is the nominal rate per annum which, if paid semi-annually, is equivalent to an effective rate of .99505% per quarter?

(39) (a) If the nominal rate is 4% per annum, payable semi-annually, what nominal rate per annum, payable quarterly, will produce the same income?

(b) What is the equivalent nominal annual rate, payable monthly?

(40) Interest being 6% per annum, payable quarterly (the effective rate per annum being therefore 1.015^4), which is the more valuable—an income of $4,080, payable at the end of the year, or an income of $4,000, of which $1,000 is payable at the end of each quarter?

(41) Interest being worth 5% per annum converted

* In connection with the text of Chapter VIII.

quarterly, what rate should be paid annually as an equivalent? (Note that the expressions "payable annually," "payable quarterly," etc., signify—through custom—that the interest is payable at the *end* of the year, quarter, etc. When interest is paid before the end of the interest period, an element of discounting enters in.)

(42) (a) Given 5% as the effective annual rate; describe the process of finding the effective quarterly rate equivalent thereto.

(b) What is the quarterly rate so found?

(c) To what *nominal* annual rate is this quarterly rate equivalent?

(43) A note for $1,000, due in one year, is discounted at the beginning of the term, the net proceeds being $940:

(a) What is the discount rate?

(b) What is the interest rate which is actually being paid?

(44) If the above note were for six months and the net proceeds were $970, what would be the nominal annual interest rate?

(45) Suppose the above note were for three months and the net proceeds $985; find the nominal annual interest rate.

§ 237. Answers to Problems in Nominal and Effective Rates

Problem (36)
12.68%

Problem (37)
12.36%

Problem (38)
4%

Problem (39)
(a) 3.98%
(b) 3.97%

Problem (40)
The latter, by $10.90

Problem (41)
5.095%

Problem (42)
(a) Find the 4th root of 1.05.
(b) 1.2272%
(c) 4.9088%

Problem (43)
(a) 6%
(b) 6.383%

Problem (44)
6.186%

Problem (45)
6.091%

§ 238. Constant Compounding

In § 93 it was stated that if an investment on a 6% nominal annual rate were compounded every millionth of a second, or constantly, the effective annual rate could never be so great as 6.184%. It may be interesting to know how to ascertain this limit. The following rule gives the method:

Rule: Multiply the constant quantity .4342944819 +, or so much thereof as is necessary, by the nominal rate per annum expressed decimally; find the logarithm of the product; from this logarithm, subtract 1, and the remainder is the effective annual rate required.

For example, take a 6% nominal annual rate. .4342944819 × .06 = .026057668914. But this latter number is the

logarithm of 1.061837, which, diminished by 1, gives .061837, which is the limit required.*

§ 239. Finding Nominal Rate

The opposite rule for finding a nominal rate which, if compounded an infinite number of times, will amount to a given effective rate at the end of the year, is as follows:

Rule: Multiply the logarithm of the effective ratio by the constant quantity 2.302585092994 +, or so much thereof as is necessary, and the product will be the nominal rate itself.†

Example: What rate compounded continuously will amount to an effective rate of 6%? Log. 1.06 = .02530587; this multiplied by 2.302585 gives .058270, the rate required.

§ 240. Approximate Rules

An approximation to the rate may also be obtained by

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* For the benefit of more advanced readers, an algebraic demonstration of the rule is here given:

$\left(1+\frac{.06}{n}\right)^n = e^{.06}$, when n becomes infinite.

log. $e^{.06}$ = .06 log. e = .06 log. 2.7182818284 = .06 (.4342944819) = .026057668914 = log. 1.061837. Therefore, $e^{.06}$ = 1.061837.

Therefore, $\left(1+\frac{.06}{n}\right)^n = 1.061837$, when n becomes infinite.

The quantity e, used above, is the base of the Napierian system of logarithms and is the sum of the infinite series,

$$1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\ldots\ldots\ldots$$

---

† An algebraic demonstration of the rule is as follows:

If $\left(1+\frac{x}{n}\right)^n = 1.06$, when n becomes infinite, find the value of x, i.e., the nominal rate.

$\left(1+\frac{x}{n}\right)^n = e^x$, when n becomes infinite; or $e^x = 1.06$.

Therefore, x(log. e) = log. 1.06

Therefore, $x = (\log 1.06)\left(\frac{1}{\log. e}\right) = (\log. 1.06)\left(\frac{1}{.4342944819}\right)$

$= (\log. 1.06)\ (2.302585092994).$

subtracting from the rate half its square. The square of .06 is .0036, one-half of which is .0018; .06 — .0018 = .0582.

Another approximation may be obtained by taking the

mean between the effective interest rate..........	.06
and the corresponding discount rate..............	.0566
which, added together, give.....................	.1166
Half of this is the approximate nominal rate......	.0583

CHAPTER XXI

EQUIVALENT RATES OF INTEREST—BOND VALUATIONS

§ 241. Annual and Semi-Annual Interest

The great majority of investments pay interest semi-annually. Occasionally annual-interest securities are offered, and it will be useful, for comparison with the ordinary semi-annual securities, to know the equivalent rates. The following table shows the equivalents for the more common annual rates, the decimals being carried to the nearest one-thousandth of one per cent.

TABLE OF EQUIVALENT RATES OF INTEREST PAYABLE ANNUALLY AND SEMI-ANNUALLY

Nominal Rate Per Annum, Payable Annually			Nominal Rate Per Annum, Payable Semi-annually
2.50%	equivalent	to	2.485%
2.55%	"	"	2.534%
2.60%	"	"	2.583%
2.65%	"	"	2.633%
2.70%	"	"	2.682%
2.75%	"	"	2.731%
2.80%	"	"	2.781%
2.85%	"	"	2.830%
2.90%	"	"	2.879%
2.95%	"	"	2.929%

Nominal Rate Per Annum, Payable Annually *(Continued)*			Nominal Rate Per Annum, Payable Semi-annually *(Continued)*
3.00%	equivalent	to	2.978%
3.05%	"	"	3.027%
3.10%	"	"	3.076%
3.15%	"	"	3.126%
3.20%	"	"	3.174%
3.25%	"	"	3.224%
3.30%	"	"	3.273%
3.35%	"	"	3.322%
3.40%	"	"	3.372%
3.45%	"	"	3.421%
3.50%	"	"	3.470%
3.55%	"	"	3.519%
3.60%	"	"	3.568%
3.65%	"	"	3.617%
3.70%	"	"	3.666%
3.75%	"	"	3.715%
3.80%	"	"	3.765%
3.85%	"	"	3.814%
3.90%	"	"	3.863%
3.95%	"	"	3.912%
4.00%	"	"	3.961%
4.05%	"	"	4.010%
4.10%	"	"	4.059%
4.15%	"	"	4.108%
4.20%	"	"	4.157%
4.25%	"	"	4.206%
4.30%	"	"	4.255%
4.35%	"	"	4.304%

Nominal Rate Per Annum, Payable Annually (Continued)			Nominal Rate Per Annum, Payable Semi-annually (Continued)
4.40%	equivalent	to	4.353%
4.45%	"	"	4.402%
4.50%	"	"	4.450%
4.55%	"	"	4.500%
4.60%	"	"	4.548%
4.65%	"	"	4.597%
4.70%	"	"	4.646%
4.75%	"	"	4.695%
4.80%	"	"	4.744%
4.85%	"	"	4.793%
4.90%	"	"	4.841%
4.95%	"	"	4.890%
5.00%	"	"	4.939%
5.25%	"	"	5.183%
5.50%	"	"	5.426%
5.75%	"	"	5.670%
6.00%	"	"	5.913%
6.25%	"	"	6.155%
6.50%	"	"	6.398%
6.75%	"	"	6.640%
7.00%	"	"	6.882%

As an illustration of the use of the above table, take the annual rate 2.50%. In this case, the square of 1.012425, which is the semi-annual effective ratio, equals approximately 1.025, the annual ratio of increase. In the case of the annual rate 4.45%, the square of 1.02201 equals approximately 1.0445, etc.

§ 242. Semi-Annual and Quarterly Interest

Quarterly bonds also occur, but with less frequency than semi-annual bonds. Some companies, in order to induce holders of bonds to register them, pay interest quarterly after registration, but semi-annually while in coupon form. Sometimes, therefore, it is desirable to know approximately how much improvement in income will result from the quarterly payments.

TABLE OF EQUIVALENT RATES OF INTEREST PAYABLE SEMI-ANNUALLY AND QUARTERLY

Nominal Rate Per Annum, Payable Quarterly			Nominal Rate Per Annum, Payable Semi-annually
2.50%	equivalent	to	2.508%
2.55%	"	"	2.558%
2.60%	"	"	2.608%
2.65%	"	"	2.659%
2.70%	"	"	2.709%
2.75%	"	"	2.759%
2.80%	"	"	2.810%
2.85%	"	"	2.860%
2.90%	"	"	2.910%
2.95%	"	"	2.961%
3.00%	"	"	3.011%
3.05%	"	"	3.062%
3.10%	"	"	3.112%
3.15%	"	"	3.162%
3.20%	"	"	3.213%
3.25%	"	"	3.263%
3.30%	"	"	3.314%
3.35%	"	"	3.364%
3.40%	"	"	3.414%

Nominal Rate Per Annum, Payable Quarterly (Continued)			Nominal Rate Per Annum, Payable Semi-annually (Continued)
3.45%	equivalent	to	3.465%
3.50%	"	"	3.515%
3.55%	"	"	3.566%
3.60%	"	"	3.616%
3.65%	"	"	3.667%
3.70%	"	"	3.717%
3.75%	"	"	3.768%
3.80%	"	"	3.818%
3.85%	"	"	3.869%
3.90%	"	"	3.919%
3.95%	"	"	3.970%
4.00%	"	"	4.020%
4.05%	"	"	4.071%
4.10%	"	"	4.121%
4.15%	"	"	4.172%
4.20%	"	"	4.222%
4.25%	"	"	4.273%
4.30%	"	"	4.323%
4.35%	"	"	4.374%
4.40%	"	"	4.424%
4.45%	"	"	4.475%
4.50%	"	"	4.525%
4.55%	"	"	4.576%
4.60%	"	"	4.626%
4.65%	"	"	4.677%
4.70%	"	"	4.728%
4.75%	"	"	4.778%
4.80%	"	"	4.829%
4.85%	"	"	4.879%

Nominal Rate Per Annum, Payable Quarterly *(Continued)*			Nominal Rate Per Annum, Payable Semi-annually *(Continued)*
4.90%	equivalent	to	4.930%
4.95%	"	"	4.981%
5.00%	"	"	5.031%
5.25%	"	"	5.284%
5.50%	"	"	5.538%
5.75%	"	"	5.791%
6.00%	"	"	6.045%
6.25%	"	"	6.299%
6.50%	"	"	6.553%
6.75%	"	"	6.807%
7.00%	"	"	7.061%

In illustration of the above table, take the rate 4.20% given in the first column. The quarterly ratio is then 1.0105. The square of this is 1.02111025, which is the semi-annual equivalent earning ratio; the equivalent semi-annual rate is 2.111025%, and the nominal annual rate equivalent to the last-named figure is approximately 4.222%.

§ 243. Problems in Valuation of Bonds*

In the following problems, all bonds are supposed to be semi-annual, unless otherwise stated.

(46) What is the difference between the cash and income rates of:

* In connection with the text of Chapter X.

(a) 4% bond netting $2\frac{1}{2}$%
(b) 3% bond netting $2\frac{1}{2}$%
(c) 5% bond netting 3.40%
(d) 3% bond netting 3.40%
(e) 7% bond netting 4%
(f) 5% bond netting 4.80%
(g) 3.65% bond netting 5%

(47) Remembering that the premium or discount on a bond is the present worth of an annuity of the difference in rates, at the income rate, and that problems have already been given involving the computation of present worths at the foregoing income rates (Problem 26), find the premium or discount on the following bonds, and hence their value, par being $1,000 in each case:

(a) 4% bond netting $2\frac{1}{2}$%, 15 years
(b) 3% bond netting $2\frac{1}{2}$%, 15 years
(c) 5% bond netting 3.40%, 25 years
(d) 3% bond netting 3.40%, 25 years
(e) 7% bond netting 4%, 10 years
(f) 5% bond netting 4.80%, 34 years
(g) 3.65% bond netting 5%, 35 years

§ 244. Successive Method of Bond Valuation—Problems

By adding the net income for one period to each of the computed values, and subtracting the cash interest, find the next periodic value at $14\frac{1}{2}$, $24\frac{1}{2}$, $9\frac{1}{2}$, $33\frac{1}{2}$, and $34\frac{1}{2}$ years, respectively. Continue this operation as many times as you please, and at any point you may prove your work by a fresh computation of the annuity.

(48) Find the value of a $4\frac{1}{2}$% bond having a par of $10,000, netting $3\frac{1}{2}$%, and having three years to run. From this initial value, work out the values successively down to par at maturity, and construct a schedule as in § 122.

(49) Perform the same operation with:

(a) a 4% bond
(b) a 3% bond
(c) a 2% bond

(50) By the use of logarithms, find the values of the following bonds of $1,000 each:

(a) 4% bond, netting 4.50%, 95 years
(b) 3½% bond, netting 3%, 40½ years
(c) 7% bond, netting 4½%, 45 years
(d) 5% bond, netting 4%, 28 years
(e) 3½% bond, netting 3.80%, 100 years

§ 245. Answers to Bond Valuation Problems

Problem (46)

(a) .75%	(c) .80%	(e) 1.5%
(b) .25%	(d) .20%	(f) .1%
		(g) .675%

Problem (47)

(a) $1,186.67	(c) $1,268.01	(e) $1,245.27
(b) $1,062.22	(d) $933.00	(f) $1,033.36
		(g) $777.94

Problem (48)

$10,282.45

Problem (49)

(a) $10,141.22 (b) $9,858.78 (c) $9,576.33

Problem (50)

(a) $890.51	(c) $1,480.56	(e) $922.88
(b) $1,116.77	(d) $1,167.52	

§ 246. Bond Valuations by the Use of Logarithms

The following will illustrate the method of solution by logarithms, taking (for example) Problem (50-a). Here the number of periods is 190, the difference between the

cash and income rates per period is \$2.50, and the income rate is 2.25% per period. We must therefore find the present worth (P) of an annuity of \$2.50 for 190 periods at 2.25%, and subtract this from the par of the bond (\$1,000), since this bond is at a discount, the income rate being larger than the cash rate. The formula for the value of the discount on a bond, as given in § 159,

is $(i-c)\left(\dfrac{1-\dfrac{1}{(1+i)^n}}{i}\right)$

which becomes $(2.50)\left(\dfrac{1-\dfrac{1}{1.0225^{190}}}{.0225}\right)$

Now, log. $1.0225 = .00966331668$.

Therefore, log. $1.0225^{190} = 190 \times .00966331668 =$
1.8360301692.

Hence, log. $\left(\dfrac{1}{1.0225^{190}}\right) = \text{log. } 1 - \text{log. } 1.0225^{190}$

$= \text{zero} - 1.8360301692 = \bar{2}.1639698308$.

The number corresponding to this logarithm is .014587128.

The value of the discount thus becomes:

$$2.50\left(\frac{1-.014587128}{.0225}\right)$$

which equals \$109.49. This discount when deducted from the par of \$1,000 gives the value of the bond, \$890.51.

The solution by logarithms involves considerable "figuring," but is nevertheless far superior to any solution by ordinary arithmetic. The labor of finding the present worth of an annuity for 190 periods by arithmetic would be intolerable.

§ 247. Finding Initial Book Values

The methods of finding the initial book values of the bonds in Schedules (A) and (B) (§ 122) are not shown in the text. The operation is here given without logarithms, and with some variations in method.

Take the case of the bond in Schedule (A), a 5% bond for $100,000 to net 4%, due in 5 years. The problem is to find the present value of an annuity of $500 for 10 periods at the ratio 1.02; but in the present method we also require the separate present worths of each instalment of $500. These ten present worths are the ten respective amounts of amortization for the ten periods in the life of the bond.

The present worth of the first instalment of $500 (i.e., the first amortization) will be $\$500 \div 1.02^{10}$; the present worth of the second will be $\$500 \div 1.02^{9}$; etc. Since multiplication is easier than division, it will be best to obtain first the value of $1 \div 1.02^{10}$; 500 times this will give the present worth of the first instalment, or the first entry in the amortization column. From the first amortization, the second and following ones may be obtained by successive multiplications by 1.02.

To obtain the value of $1 \div 1.02^{10}$, we find first the 10th power of 1.02. After multiplying 1.02 by itself, we do not again use it as a multiplier, but square the square, giving the fourth power. The 4th multiplied by the 4th gives the 8th, and the 8th multiplied by the 2nd gives the 10th power of 1.02, as shown on the following page. A check on the accuracy of the result may also be obtained by employing the method suggested in the footnote of § 19. In this latter case, the process consists in finding the value of $(1 + .06)^{10}$ by the use of the algebraic formula known as the binomial theorem.

```
 102
 102
 ───
 102
   204
 ─────
 10404            (102²) (It is unnecessary to
   41616             repeat the multiplier)
     41616
 ─────────
 108243216               (102⁴) = (102²)²
   865945728
     21648643 ||  (contracted multiplication)
      4329729 ||
       324730 ||
        21649 ||
         1082 ||
          649 ||
 ────────────
 11716593810 ||  (102)⁸ = (102⁴)²
 10404
 ────────────
 11716593810 ||
   468663752 ||
     4686638 ||
 ────────────
 12189944200   (102)¹⁰ = (102⁸) × (102²)
```

§ 248. Tabular Multiplication and Contracted Division

Next, 1 is to be divided by 1.21899442. We shall use contracted multiplication, and further facilitate the work by employing the tabular plan. This consists in preparing in advance a table of the first 9 multiples of 1.21899442 in such a way that we are certain of their correctness. The use of a table such as this greatly facilitates accuracy and quickness in performing the division of several numbers by the same divisor, especially in cases where the divisor is lengthy and no calculating machines are available.

On the first line of the table we set down the number, and on the second line, its double.

1	121899442
2	243798884
3	
4	
5	
6	
7	
8	
9	
Proof	

The third line is formed by adding the first to the second, and all the others in succession by adding the first. The proof line is 10 times the original, if there is no mistake in the work.

1	121899442
2	243798884
3	365698326
4	487597768
5	609497210
6	731396652
7	853296094
8	975195536
9	1097094978
Proof	1218994420

The contracted division consists in merely subtracting these multiples. The quotient may as well be placed *above* the dividend to save space.

Quotient	820348300
Dividend	1000000000
(8)	975195536
	24804464
(2)	24379888‖
	424576
(03)	365698‖
	58878
(4)	48760‖
	10118
(8)	9752‖
	366
(3)	366‖

\$.8203483 is therefore the present worth of \$1 due in 5 years; its product by 500 is the first amortization:

	\$410.17415	
Subtracting this from..........	500.	
gives the compound discount....\$	89.82585	
Dividing this by .02 gives.......	4491.2925	$(D \div i = P)$
or the premium, rounded to......	4491.29‖	

§ 249. Formation of Successive Amortizations

Our amortization column will begin with \$410.17, and each successive term will be 1.02 times the preceding, while the sum of the column must be \$4,491.29. To insure accuracy in the last figure, it will be well to retain at least the mills. Having obtained all the ten terms, the multiplication is performed once more, giving as a test \$500. The terms

are again tested by addition, bringing the result, \$4,491.29. Then the book values beginning with \$104,491.29, and ending with \$100,000, are formed by subtraction, still retaining the mills. In making up the schedule the values are rounded to the nearest cent, and the amortization column is made to correspond.

	\$104,491.292
\$410.174	410.174
	\$104,081.118
418.377	418.377
	\$103,662.741
426.745	426.745
	\$103,235.996
435.280	435.280
	\$102,800.716
443.986	443.986
	\$102,356.730
452.866	452.866
	\$101,903.864
461.923	461.923
	\$101,441.941
471.161	471.161
	\$100,970.780
480.584	480.584
	\$100,490.196
490.196*	490.196
	\$100,000.000
Total, \$4,491.292	

* \$490.196 × 1.02 = \$500

§ 250. Test by Differencing

In a successive computation like the one just given, a slight error increases at every step, and there is danger that a great many terms may have to be recalculated. The method of *differencing,* applied during the progress of the work, will form an efficient check on all except the last figure.

§ 251. Successive Columns

To difference a series, we first set down its terms in a first column. In the second column we set down the first differences (D_1), of which the first line is the difference between the first term and the second, the second line is the difference between the second and the third, and so on. D_2 is composed of the differences between these first differences. D_3 is formed from D_2 in just the same way as D_2 from D_1, and all succeeding differences in the same way, to the extent required.

The terms just obtained in amortizing $104,491.292 down to par, would be differenced as follows:

Term	D_1	D_2	D_3
410.174	8.203	.165	.002
418.377	8.368	.167	.004
426.745	8.535	.171	.003
435.280	8.706	.174	.003
443.986	8.880	.177	.004
452.866	9.057	.181	.004
461.923	9.238	.185	.004
471.161	9.423	.189	.003
480.584	9.612	.192	
490.196	9.804		
500.000			

§ 252. Intentional Errors

To demonstrate the utility of the method, introduce an error purposely by altering one of the figures in a term at least three or four lines from the top. Even a mill, when all the differences are carried out, will cause violent fluctuations in the column D_3 and instantly call attention to the error.

§ 253. Rejected Decimals

The reason the fourth column shows some fluctuation even though no errors have been made, is that the last figure of a term is never accurate, but always rounded off or up. In a third difference-column, this residue of error increases threefold; in a fourth column, it may reach six times the original rounding, and, in the fifth, ten times.

§ 254. Limit of Tolerance

The extent to which the last column of differences may be allowed to "waver" will be learned by experience. The next-to-the-last column should be progressive; that is, it should never change its course and go backward; it should either constantly increase or constantly decrease.

It will be a useful exercise to take the more extended value, $410.17415 (instead of $410.174), multiply it up to $500, and difference the results out to 5 differences. A very minute error will become enormously magnified and call attention to itself.

CHAPTER XXII

BROKEN INITIAL AND SHORT TERMINAL BONDS

§ 255. Problems in Valuation*

(51) Suppose the value of a 4% bond for 15 years on a $2\frac{1}{2}$% basis to be, as shown in Problem (47-a), \$1,-186.66680; what would be its value one month later, the time prior to maturity then being 14 years, 11 months? (Since we are dealing with half-years, this time must be treated as $14\frac{1}{2}$ years, 5 months, or 15 years less 1/6 of the semi-annual amortization period.)

The theoretical, or mathematically correct, value (§ 129) in the above case would be ascertained as follows:

The ratio of increase is..............	1.0125
Its logarithm is.....................	.005 395 031 887
This must be divided by 6, giving.....	.000 899 171 981
which is the logarithm of the 6th root of 1.0125, or (in other words) the logarithm of the effective ratio for 1/6 of a semi-annual period.	
The number corresponding to the last logarithm is	1.002 072 564 8
Multiplying the value at the beginning of the 15-year period (\$1,-186.66680), by this number, gives the flat value at 14 years, 11 months, before maturity	\$1,189.126 21

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* In connection with the text of Chapter XI.

Although the above method is never used in actual buying or selling, yet it is proper for estimating results of financial operations.

(52) A firm of brokers offers \$50,000 of 3% bonds, due July 1, 1929, J & J, on a 2½% basis. What should be the price on September 25, 1914, flat or "and interest"?

(53) On July 10, 1913, \$25,000 of 5% bonds due April 1, 1938, A & O, are bought at a price to yield 3.40%.

(a) What is the flat price?

(b) What is the price "and interest"?

(54) \$10,000 of 3% bonds due January 1, 1938, J & J, are purchased to net 3.40%. Find the price exclusive of interest and the price flat, on May 16, 1913.

(55) \$6,000 of 4½% bonds were issued in 1908, due April 1, 1928, M & N. Find the price "and interest," on July 1, 1914, on a 4.80% basis.

(56) An investor owns the four lots of bonds mentioned in Problems (52), (53), (54), and (55), and has hitherto carried them on his books at par. He desires to have them adjusted to investment value as of December 31, 1914. What will be the investment value:

(a) Of each lot?

(b) Of the aggregate?

(57) Find the amount of amortization for the semi-annual period ending June 30, 1915:

(a) On each of these lots of bonds.

(b) On the aggregate of the four lots.

(58) Taking the bonds in Problem (53), ascertain their values at April 1 and October 1, 1937, and thence at July 1, 1937. From this last value, (a) amortize to January 1, 1938, (b) and then for the broken period to April 1, 1938, when they should reduce to par.

(59) Taking the bonds in Schedule (H) (§ 141), re-

construct the schedule so that the next date after May 1, 1914, is July 1, 1914; then January 1, 1915, and so on at balancing periods, giving a J & J schedule instead of an M & N schedule.

(60) A certain issue of $100,000 of 4% bonds is dated September 1, 1913, and interest begins at that date; but interest is payable on February 1 and August 1, and the principal (with 4 months' interest) is payable December 1, 1917.

(a) What is the value of the bonds on a 3.60% basis at the date of issue?

(b) What is their value on the same basis if purchased at December 1, 1913?

(c) At August 1, 1917?

(In this question, note that the period at the beginning is for 5 months, and not the usual 6 months.)

(61) Make a schedule running from December 1, 1913, to maturity, of the above bonds at the F & A dates.

(62) Make a schedule as above, but with J & J dates, for balancing purposes.

(63) A broker offers the above bonds on December 1, 1913, at 101.50 (meaning $101.50 for each $100 of par, which is the customary phrase), which he says will pay *about* 3.60%. Eliminate any residue by the methods in §§ 136 to 139, inclusive, making a J & J schedule running to maturity.

As will be noted, this last example contains all of the following peculiarities: short initial period, odd purchase date in that period, short terminal period, interpolated balance dates, and residue to be eliminated.

§ 256. Answers to Valuation Problems

Problem (51)

$1,189.13902 (by the customary method).

Problem (52)

$53,420.93 flat, or $53,070.93 and interest.

Problem (53)

(a) $31,996.64 flat.
(b) $31,652.89 and interest.

Problem (54)

$9,336.43 and interest, or $9,448.93 flat.

Problem (55)

$5,820.34 and interest.

Problem (56)

(a) $53,025.00; $31,392.26; $9,365.30; $5,825.03.
(b) $99,607.59.

Problem (57)

Amortization, $87.19 and $91.33; accumulation, $9.21 and $4.80; net amortization, $164.51.

Problem (58)

Value at January 1, 1938, $25,098.33; for the broken period from January 1 to April 1, 1938, interest on premium is $1.67, interest on par is $212.50, and cash interest is $312.50, thus reducing the bond to par.

Problem (59)

July 1, 1914, $104,693.02; January 1, 1915, $104,-286.88; etc.; July 1, 1919, $100,245.90.

Problem (60)

(a) September 1, 1913, $101,563.90.
(b) December 1, 1913, $101,477.98.
(c) August 1, 1917, $100,131.75.

Problem (61)

Value at February 1, 1914, $101,420.69; etc.

Problem (62)

Value at January 1, 1914, $101,449.33; at July 1, 1914, $101,275.34; etc.

Problem (63)

The residue is $22.02, being the difference between $101,500.00 and $101,477.98. The coefficient for elimination of the residue is 1.0148987, meaning that for every dollar of amortization on the bonds bought at the *exact* 3.60% basis, there should be added 1.48987c. if the bonds are bought on the *approximate* 3.60% basis, i.e., $101,500.00.

CHAPTER XXIII

THE USE OF TABLES IN DETERMINING THE ACCURATE INCOME RATE

§ 257. Bond Tables as Annuity Tables

The "Extended Bond Tables"* can be used as an annuity table in case of need, when the latter is not at hand or when the figures in it are not sufficiently extended or the rates not sufficiently close.

In using the "Extended Bond Tables" for this purpose, it must be remembered that its results are based on semi-annual payments of interest, the periods being half-years. In the foregoing problems on annuities where periods and rates per period are used, in order to make use of the bond tables these "periods and rates per period" must be transformed into years and rates per annum, payable semi-annually. In this manner the data given in Problem (26) will be changed as follows:

1.25%, 30 periods, becomes 2.50%, 15 years.
1.70%, 50 periods, becomes 3.40%, 25 years.
2.00%, 10 periods, becomes 4.00%, 5 years.
2.40%, 68 periods, becomes 4.80%, 34 years.
2.50%, 70 periods, becomes 5.00%, 35 years.

§ 258. Premium and Discount as a Present Worth

As explained in Chapter X, the premium or discount on a bond is nothing more or less than the present worth, at

* Sprague's "Extended Bond Tables."

the income rate, of an annuity for the life of the bond equal to the difference between the cash and income rates. Taking, as an illustration, the second case mentioned above, 3.40% for 25 years, we turn to the 5% bond table, page 88,* and find the value of such a bond to be......$1,268,009.70
The value of a similar bond in the 4% table,

page 54* is.......................... 1,100,503.64

Difference $167,506.06

The first amount results from a cash rate of 5% and an income rate of 3.40%; in the case of the second amount, the cash rate is 4%, with the same income rate. The difference between these two amounts arises therefore on account of the difference in cash rates, which, for a bond of $1,000,000, is $10,000 annually. In other words, the difference is the present worth of an annuity of $10,000 per annum, payable semi-annually, at 3.40% for 25 years. Expressed in periods, it is the present worth of an annuity of $5,000 per period, for 50 periods, at 1.70% per period. The present worth of an annuity of $1 per period, under like conditions, would therefore be 1/5000 of $167,506.06, or $33.501212.

§ 259. Present Worth by Differences

Instead of using the coupon rates 4% and 5%, we might have selected 3% and 4%, 3½% and 4½%, 5% and 6%, or any other two rates differing by 1%. For example:

Value of 4% bond, yielding 3.40%.........$1,100,503.64
Value of 3% bond, yielding 3.40%.......... 932,997.57

Difference, being value of annuity........... $167,506.07

* Sprague's "Extended Bond Tables."

There is a discrepancy of one cent in comparison with the previous difference, owing to the rounded decimals.

The reason for the process may be explained as follows:

On a 5% bond of \$1,000,000 yielding 3.40% the cash, or coupon, interest is	\$50,000	
the net income is	34,000	
Difference		\$16,000
In the case of a 4% bond, the interest at the cash or coupon rate is	\$40,000	
the net income is	34,000	
Difference		\$6.000

The term "net income," as here used, has a slightly different meaning from its use in the schedules in Chapters X and XI; in the latter case, the income rate was applied to the book value, while in the present instance it is applied to the par value.

Hence, from the bond tables we may derive the present worths of two annuities of \$16,000 and \$6,000 (being respectively \$268,009.70 and \$100,503.64), and their difference must always be the present worth of an annuity of \$10,000. From the foregoing, we may state the following:

Rule: The present worth of an annuity of \$10,000, payable semi-annually, at a certain income rate, is equal to the difference between the values of a 4% and a 5% bond for \$1,000,000 at the same income rate.

If it should happen that the rent of the desired annuity were \$5,000 instead of \$10,000, the present worth thereof might be obtained at once from the difference in values between 3% and 3½% bonds, or between 3½% and 4% bonds. Similarly, the difference between 3½% and 5% bonds would give the present worth of an annuity of \$15,-

000; 3% and 5%, \$20,000; 3½% and 6%, \$25,000; 3% and 6%, \$30,000; 3½% and 7%, \$35,000; and 3% and 7%, \$40,000. These results would be a trifle more accurate in the last figure than those obtained by multiplying the present worths of the \$10,000 annuities, since the multiplication of figures which have been rounded increases the error.

§ 260. Present Worth by Division

The present worth of an annuity may also be obtained by division from a single bond value, instead of taking the difference between two. We saw that the premium on a 4% bond to net 3.40% is the present worth of an annuity of \$6,000, payable semi-annually; therefore, if the premium be divided by 6, it will give the present worth of an annuity of \$1,000, payable semi-annually:

$$\$100{,}503.64 \div 6 = \$16{,}750.61$$

§ 261. Compound Discount and Present Value of a Single Sum

From the present worth of an annuity of \$10,000 obtained as above, the compound discount and the present value of a single sum for the same time and rate can also be ascertained. Multiplying the present worth of the annuity by the number of units in the rate per cent gives the compound discount on a single sum of \$1,000,000.

\$167,506.06 times 3.4 =	\$569,520.60	compound discount
Subtract this from.....	1,000,000.00	
and we have..........	\$430,479.40,	which is the present

worth of \$1,000,000 payable in a single sum in 25 years at 3.40% compounded semi-annually. These computations are merely applications of the two formulas, $P \times i = D$ (§ 67) and $p = 1 - D$ (§ 35). The last figure in the above present

worth is unreliable; as a matter of fact, the cents should be 38.

If necessary, in the absence of compound interest tables or logarithms, the amount of a single sum at compound interest may be obtained through the application of the formula $a = 1 \div p$ (§ 35). The present value of \$1 (or p) is \$.4304794; therefore, divide 1 by .4304794, using contracted multiplication.

```
4304794 ) 1.0000000 ( 2.3229917
           8609588
          --------
           1390412
           1291438||  (The sign || indicates con-
          --------      traction or rounding.)
             98974
             86096||
            ------
             12878
              8610||
              ----
              4268
              3874||    (For explanation of
              ----      contracted multiplica-
               394      tion, see § 228.)
               387||
              ----
                 7
                 4||
                 -
                 3
```

§ 262. Use of Bond Tables in Compound Interest Problems

The amount of \$1 for 50 periods at 1.70% per period, as above computed, is \$2.3229917, and the compound interest is \$1.3229917. If the latter amount be divided by .017, the rate of income for a single period, the result (\$77.82306) will be the amount of an annuity of \$1 for 50 periods at

1.70%; being an application of the rule (§ 60) $A = 1 \div i$. Again, when this result ($77.82306, or A) is divided by 2.3229917 (or a) the quotient is $33.5012 + (or P), which is the present worth of an annuity of $1 for 50 periods at 1.70%. This is an application of the formula (§ 67) $P = A \div a$. The quotient last obtained checks very closely with the result previously found for the value of P, $33.501212. Thus, all of the problems in compound interest are soluble through the bond tables.

§263. Determination of the Accurate Income Rate*

As stated (§ 136), values of bonds for each one-hundredth of one per cent of gradation in the ordinary income rates may be obtained from Sprague's "Extended Bond Tables." If, however, an even more minute degree of accuracy is desired in the income rate, as, for example, a rate like 4.2678%, these tables are not sufficient. In order to develop a method to accomplish this result, we will first state the problem in symbolic form:

Given a bond on which there is a premium or discount Q, cash rate c, and number of periods n, what is the income rate i?

Every premium or discount is the present worth, at the income rate, of an annuity of n terms, each instalment of which is the difference between the cash and income rates; in other words, it is the present worth of an annuity of $1 multiplied by the difference in rates (§ 118). Writing P for the present worth of an annuity of $1, we have the equation: $Q = P \times (c - i)$. The terms c and i, in the great majority of bonds, *theoretically* refer to the rates for *semi-annual* periods. *In practice,* however, a 4% rate or a 5% rate means an *annual* nominal rate, irrespective of the fact that the coupons are semi-annual. In order to conform to

---

* Compare text of §§ 135, 136.

commercial usage, we will alter the equation by halving the P and doubling the $(c—i)$; the equation then becomes: $Q = \frac{1}{2}P \times (2c—2i)$. With this change, the value of the right-hand member is not altered, and there is the advantage that the quantities $2c$ and $2i$ represent, respectively, the nominal annual cash and income rates.

§ 264. Assumed Trial Rate

In the equation given above, the premium or discount Q is known, and the cash rate c is also known. There is therefore, in reality, but one unknown quantity, the income rate i, since P can be ascertained when once the value of i is known. It is evident that if we divide Q by $\frac{1}{2}$P (which latter we will hereafter call the trial divisor), we shall find the difference in rates. Let us *assume* the rate of income to be any rate whatever, and then calculate the trial divisor at that rate. Then, since the product of $\frac{1}{2}$P times $(2c—2i)$ is the constant, or known, quantity Q, we have the following chain of reasoning: If the assumed income rate is too small, P will be too large, the difference in rates will be too small, and the ascertained income rate will be too large; and *vice versa* if the assumed income rate is too large. Taking now this first ascertained rate as the new assumed rate, we may find a second ascertained rate, and so on, as many times as we please, the proceeding being something like the swinging of a clock pendulum, except that each swing is shorter than the preceding one, since the successive ascertained rates, one after another, more nearly approach the true income rate. We may slightly modify any rate in order to make the work easier; if we are fortunate in selecting our first trial rate near the true rate, fewer successive approximations will be necessary.

For the purpose of computing the value of the trial divisor ($\frac{1}{2}$P), a table of bond values may be used for

the first two or three approximations, by taking the difference between the values (at the same income rate) of a 3% bond and a 4% bond, or of some other pair of bonds whose nominal annual cash rates differ by 1%.

§ 265. Application of Assumed Trial Rate—Bond Above Par

As an example, we will take a 6% semi-annual bond for \$100, due in 50 years and sold at 133, to find the income rate. With so large a premium as 33, the income rate is evidently much less than 6%; let us assume 4%. From the bond tables we find that the value of a 5% bond, due in 50 years, and earning 4%, on a par of 100, is....\$121.549

The value of a similar bond earning only 4% is par, or	100.000
The difference is the present worth of an annuity of 50c. (the difference between the semi-annual cash and income rates) for 100 periods at 2% per period	\$21.549
The present worth of a similar annuity of \$1, or P, is	\$43.098
½P, the first trial divisor, is therefore	\$21.55

33.00 ÷ 21.55 = 1.531, the difference in rates. 6% — 1.531 = 4.469%, the new trial rate. Taking 4.45% as more convenient, the new trial divisor is 19.98. 33.00 ÷ 19.98 = 1.651. 6% — 1.651% = 4.349%. For this new rate (or 4.35%), we find that 20.315 is the trial divisor. 33.00 ÷ 20.315 = 1.6244. 6% — 1.6244% = 4.3756%. Next using 4.37%, the trial divisor is 20.25. 33.00 ÷ 20.25 = 4.37, almost exactly, so that the use of 4.37% as an assumed or trial rate leads to it again as an ascertained rate; in other words, the rate 4.37% reproduces itself, which shows that

we have now found the correct rate. The value of the bond at 4.37%, as computed by logarithms, is $133.0069, an error of less than one cent.

§ 266. Variations in Assumed Rates

The example in § 265 is an illustration of what we have previously pointed out; that is, that the results always swing to the opposite side of the true rate. If the trial rate is too large, the ascertained rate will be too small, and the true rate will lie between them. The successive rates were 4%, 4.469%, 4.349%,4.3756%, and 4.37%. 4.37% lies between any pair of these rates except the last two, where one rate coincides with 4.37%. The foregoing is always the case with bonds above par. With bonds below par it is different; here the true rate is always larger than the last approximation. The ascertained rate may be carried to many decimal places, but it never quite overtakes the true rate. The case is somewhat analogous to a circle having an inscribed polygon. We may increase the number of sides of the polygon indefinitely, but its area will never quite equal the area of the circle.

§ 267. Application of Assumed Trial Rate—Bond Below Par

As an example of a bond below par, take a 3% bond payable in 25 years. If purchased at 88.25, what is the income rate? The following may be the steps, the dividend being always 11.75, the discount:

Trial rates......	3.70%	3.725%	3.7265%
Trial divisors....	16.2190	16.175	16.17245
Ascertained rates.	3.7244%	3.7264%	3.7265%

Since 3.7265% reproduces itself, it must be correct to the 4th decimal. Tested by logarithms, the value of a 3% bond for 25 years yielding 3.7265% proves to be $88.25015.

§ 268. Trial Rates from Bond Tables

While the method of trial rates is correct in theory, it may be greatly facilitated in practice by first locating by means of bond tables* the required income rate between two rates one-hundredth of one per cent apart. The results will be so close that simple interpolation (explained in Chapter XXXI) will suffice for at least seven decimals, and the laborious divisions necessary in the foregoing method will be avoided.

§ 269. Use of Bond Tables

For example, let it be required to find the income rate of a 4% bond for $1,000,000 due in 100 years, bought for $1,264,806.66. From the 4% table, we find that the income rate must lie between 3.10% and 3.15%. The values corresponding to these rates are as follows:

3.10%	$1,276,929.04
3.15%	1,257,990.62

1/5 of the difference being $3,787.684, we may roughly estimate the intermediate values as follows:

3.11%	$1,273,141.35
3.12%	1,269,353.66
3.13%	1,265,565.98
3.14%	1,261,778.30

The required rate must lie between 3.13% and 3.14%; the difference in rates lies between .87% and .86%. Correcting the above intermediate values by the colored pages in the bond tables,* we have:

Premium at 3.13%	$265,505.52
Premium at 3.14%	261,738.09
Premium at the required rate	264,806.66

Since any two premiums at the same income rate are

* Sprague's "Extended Bond Tables."

directly proportional to the difference between the cash and income rates, we have the following proportions:

at 3.13% — \$265,505.52 : \$264,806.66 : : .87% : x% (x = .867709998)
at 3.14% — \$261,738.09 : \$264,806.66 : : .86% : x% (x = .870082484)

At the same premium on each bond (\$264,806.66), we see from the above two proportions that the following facts prevail with reference to the rates:

Income Rate		Cash Rate
3.13%	corresponds with	3.997709998%
3.14%	" "	4.010082484%

Our problem is to determine the income rate corresponding with a cash rate of 4%, the premium still being the same. For this purpose, the method of interpolation will be sufficiently exact, and we may form a proportion as follows:

x% : .01% :: .002290002% : .012372486%

The unknown term of the proportion is found to be .0018509%, which added to 3.13% gives 3.1318509% as the income rate corresponding to a 4% cash rate. The accurate value of the bond computed to ten decimal places at the income rate of 3.131851% is \$1,264,806.6645.

CHAPTER XXIV

DISCOUNTING*

§ 270. Table of Multiples

Discounting may be performed as well by multiplication as by division, and multiplication is preferable as being the more direct and compact process. In Table VI (§ 383) are the reciprocals of all usual ratios of increase. Multiplying by .9803921568, for example, will give the same result up to a certain number of places, as dividing by 1.02. Using the tabular plan, we have this table:

1	98039216
2	196078431
3	294117647
4	392156863
5	490196078
6	588235294
7	686274510
8	784313726
9	882352941

We will take as an illustration a 5% bond, yielding 4%, both the coupons and the income being on a semi-annual basis. The amounts receivable at maturity are $100,000.00 of principal and $2,500.00 of coupons, a total of $102,500.00. The discounting process would then be as follows:

---

* In connection with text of § 143.

102,500.00
98,039.22
1,960.78
490.20
100,490.20
2,500.00
102,990.20
98,039.22
1,960.78
882.35
88.24
.20
100,970.79
2,500.00
103,470.79
98,039.22
2,941.18 etc.

There is an error of 1 cent in the value 100,970.79; this could easily have been prevented by carrying out into mills. For long operations it is always advantageous to use a few spare places beyond those retained in the final result.

§ 271. Present Worths of Interest-Difference

Still greater brevity will be attained by working out first the items of amortization, or present worths of the difference between the cash and income rates. The present worths of the interest-difference 500 are obtained as follows, using fewer figures and less labor than in the preceding example:

500.000	
490.196	½ year before maturity
392.157	
88.235	
.098	
.088	
6	
480.584	1 year before maturity
392.157	
78.431	
.490	
.078	
4	
471.160	1½ years before maturity
392.157	
68.627	
980	
98	
59	
461.921	2 years before maturity

Writing these down in reverse order, the amortization column of the schedule is filled:

461.92
471.16
480.58
490.20
1903.86

The value at two years before maturity is therefore $101,903.86, and the schedule may be further filled:

Amortization	Book Value	Par Value
	$101,903.86	$100,000.00
$461.92	101,441.94	
471.16	100,970.78	
480.58	100,490.20	
490.20	100,000.00	

For practice, any of the Problems (52) to (55), inclusive, may be worked over backwards.

§ 272. Discounts from Tables

If the rate is one of those embraced in Table II (§ 379), and the difference of interest is a simple number, the process is still easier. Here the present worths of 500 for various numbers of periods at 2% per period are required. In Table II we find these present worths for $1; pointing off 3 places to the right gives the corresponding values for $1,000, and halving this, all in the one operation, gives the successive figures required:

4	periods	.92384543	× 1000 ÷ 2 =	461.9227||
3	"	.94232233	"	471.1612||
2	"	.96116878	"	480.5844||
1	"	.98039216	"	490.1961||
				1903.8644||

§ 273. Reussner's Tables

Reussner's "True Discount Tables" give multipliers for each day, from 1 to 180, carried to 8 places, for a great number of usual rates, and will much facilitate discounting

for fractional periods. In the example in the text, it gives .99009901 opposite 90 days at 4%, with the following result:

$$\begin{array}{r}
102{,}500.000 \\
\hline
99{,}009.901 \\
1{,}980.198 \\
495.050 \\
\hline
\end{array}$$

101,485.149, the same as in the text of § 144.

CHAPTER XXV

SERIAL BONDS

§ 274. Problem in Valuation of Serial Bonds

(64) A city issues ten 4% bonds for $10,000 each, A&O, on April 1, 1914, maturing as follows: $10,000 on April 1, 1916; $10,000 on April 1, 1918, and so on—$10,000 each alternate year, the last $10,000 on April 1, 1934. They are sold at 108.33, the purchaser believing that he has a 3.10% investment. How near right is he?

As the average time of the bonds is 11 years, it might be inferred that the true value of the series was the value of a single bond of $100,000 due in 1925, which would be $108,334.54; but this is fallacious. The true price, obtained by adding together all the separate tabular values, is always less.

At 3.10%, the values of the bond at varying due dates are as follows:

Due Dates	Values
1916	$10,173.24
1918	10,336.13
1920	10,489.31
1922	10,633.35
1924	10,768.79
1926	10,896.16
1928	11,015.92
1930	11,128.53
1932	11,234.43
1934	11,334.01
Total series	$108,009.87

It is evident that the purchaser should have paid 108.01 instead of 108.33, and that on the latter price he will earn less than 3.10%. How much less, is to be ascertained.

The value at 3.10% might have been carried out further in decimals to the limit of the tables, giving \$108,009.8686.

The values at 3.05% will next be copied down.

	Due Dates	Values
3.05% Basis	2 years	\$10,182.9714
	4 "	10,355.1945
	6 "	10,517.3006
	8 "	10,669.8840
	10 "	10,813.5041
	12 "	10,948.6875
	14 "	11,075.9297
	16 "	11,195.6973
	18 "	11,308.4294
	20 "	11,414.5391
		\$108,482.1376

§ 275. Inter-rates

The inter-rates, 3.06%, 3.07%, 3.08%, and 3.09%, can now be obtained in bulk without determining the values for separate years, according to the directions on page 123 of Sprague's "Extended Bond Tables."

Find the difference between	\$108,482.1376
and	108,009.8686
which is	\$472.2690
$\frac{1}{5}$ of this is	94.4538

Subtracting from \$108,482.1376 successively $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, and $\frac{4}{5}$, we have the approximate values for 3.06% \$108,387.6838
for 3.07% 108,293.2300

But it is unnecessary to go further; it is evident that the effective rate is a little below 3.07%.

§ 276. Table of Differences

The value given at the basis of 3.07% is approximate, and we can get a corrected value by applying the rule given on page 122 of the tables,* viz.: "To correct any terminal 2 or 7, subtract 1½ times the difference and then add 1/10 the sub-difference." The following table is derived from pages 146 to 149, inclusive, of the bond tables,* and shows the differences and sub-differences in the case of a 4% bond of $1,000,000 at the income rates of 3.05% and 3.10%.

Dates of Maturity of Bond	Differences at 3.05% Basis	Differences at 3.10% Basis	Sub-Differences
2 years	$ .09	$ .09	
4 "	.33	.33	
6 "	.69	.69	
8 "	1.17	1.16	$.01
10 "	1.74	1.73	.01
12 "	2.40	2.39	.01
14 "	3.14	3.12	.02
16 "	3.95	3.92	.03
18 "	4.82	4.78	.04
20 "	5.74	5.68	.06
Total	$24.07	$23.89	$.18

On account of the fact that each of the bonds in question has the par of $10,000 and not $1,000,000, the tabular difference for the rate 3.07% becomes $.2407, and the sub-difference $.0018; 1½ times the difference equals $.3611, and 1/10 of the sub-difference is $.0002. The corrected value at 3.07% therefore becomes:

* Sprague's "Extended Bond Tables."

$108,293.2300 — $.3611 + $.0002 =	$108,292.8691
The residue to be eliminated is......	37.1309
making the price paid......	$108,330.0000

§ 277. Successive Method

The values at the basis of 3.07% must next be worked out for each period down to the last maturity.

Value at April 1, 1914....	$108,292.8691
× 1.01535............	1,082.9287
	541.4643
	32.4879
	5.4146
less	2,000.0000
Value at October 1, 1914,	$107,955.1646
× 1.01535............	1,079.5516
	539.7758
	32.3865
	5.3978
less	2,000.0000
Value at April 1, 1915....	$107,612.2763
× 1.01535............	1,076.1228
	538.0614
	32.2837
	5.3806
less	2,000.0000
Value at October 1, 1915..	$107,264.1248

etc., etc.

At April 1, 1916, 1918, etc., at intervals of two years, the book value will be further diminished to the extent of the principal of the bonds maturing at these respective dates.

§ 278. Balancing Period

But it may be that balancing-period figures are wanted, say J & J. In that case, the value on July 1, 1914, is half-

way between.............	$108,292.8691
and	107,955.1646
or	$108,124.0168
with which we continue—	1,081.2402
	540.6201
	32.4372
	5.4062
less	2,000.0000
Value at January 1, 1915..	$107,783.7205
	1,077.8372
	538.9186
	32.3351
	5.3892
less	2,000.0000
Value at July 1, 1915....	$107,438.2006
	1,074.3820
	537.1910
	32.2315
	5.3719
less	2,000.0000
Value at January 1, 1916..	$107,087.3770

§ 279. First Payment in Series

We have now reached a point where a broken terminal period occurs, as to the first $10,000 due April 1, 1916, and we must follow the directions of § 88, with this modification: that the $10,000 and the remaining $97,087.38 must be treated separately, the reason being obvious.

	$107,087.3770
Amount of principal due April 1	10,000.0000
Remainder	$97,087.3770
The usual procedure—	
	970.8738
	485.4369
	29.1262
	4.8544
$10,000 × .007675 (3 months) ..	76.7500
	$98,654.4183
Income 2% on $90,000	
1% on $10,000 = ...	1,900.0000
Value at July 1, 1916..........	$96,754.4183

This will exemplify the process when the principal of one of the serial bonds is paid off.

§ 280. Elimination of Residue

There is a residue of $37.1309 to be eliminated, for which we shall use the third method. A total premium of $8,330 is to be amortized, while the 3.07% basis will amortize only $8,292.8691. The proportion is 8,330 ÷ 8,292.8691 = 1.0044784. A table formed from this will give the following multiples:

1	1004478
2	2008957
3	3013435
4	4017914
5	5022392
6	6026870
7	7031349
8	8035827
9	9040306

The amortization at 3.07% for the fractional period and the 4 full periods is as follows:

168.8523 (April 1 to July 1, 1914)
340.2963
345.5199
350.8236
332.9587

and as adjusted for elimination as follows, the eliminands appearing in the top line and the eliminates in the bottom line:

1688523	3402963	3455199	3508236	3329587
1004478	3013435	3013435	3013435	3013435
602687	401791	401791	502239	301344
80358	2009	50224	8036	20090
8036	904	5022	201	9040
502	63	199	36	502
23				87
1696084	3418202	3470671	3523947	3344498

As thus computed, the adjusted amounts of amortization would be as follows:

April 1, 1914, to July 1, 1914..........	$169.61
July 1, 1914, to Jan. 1, 1915..........	341.82
Jan. 1, 1915, to July 1, 1915..........	347.07
July 1, 1915, to Jan. 1, 1916..........	352.39
Jan. 1, 1916, to July 1, 1916..........	334.45
Total for 4½ years..............	$1,545.34

§ 281. **Schedule**

The schedule will then be made up as follows to this point:

Schedule of Amortization

$100,000.00 of 4% Serial Bonds of the City of..............; $10,000.00 maturing biannually at April 1, 1916, 1918, 1920, 1922, 1924, 1926, 1928, 1930, 1932 and 1934, respectively.

Date	Total Int. 4%	Net Income 3.07+%	Amortiz-ation	Book Value	Par
1914, April 1,	Cost........	....	....	$108,330.00	$100,000.00
July 1,	$1000.00	$830.39	$169.61	108,160.39	
1915, Jan. 1,	2000.00	1658.18	341.82	107,818.57	
July 1,	2000.00	1652.93	347.07	107,471.50	
1916, Jan. 1,	2000.00	1647.61	352.39	107,119.11	
July 1,	1900.00	1565.55	334.45	96,784.66	90,000.00

The premium is now \$6,784.66, and the premium at 3.07% is \$6,754.42, which we test as follows:

$$
\begin{array}{lr}
 & 1.004478 \\
 & \times\ 675442 \\
\hline
 & 602687 \\
 & 70313 \\
 & 5022 \\
 & 402 \\
 & 40 \\
 & 2 \\
\hline
\text{Proof} & 678466
\end{array}
$$

§ 282. Uneven Loans

The terms of a series of bonds need not necessarily be of like amount. Suppose the payments in the above example were:

\$10,000	in 1916
\$20,000	in 1918
\$30,000	in 1920
\$40,000	in 1922
\$100,000	

and it were desired to find the value at 3.10%; the process would be:

\$10,173.2358		\$10,173.2358
10,336.1340 × 2		20,672.2680
10,489.3124 × 3		31,467.9372
10,633.3506 × 4		42,533.4024
	Value of series	\$104,846.8434

The formation of the schedule would be precisely analogous to that already given.

§ 283. Tabular Methods

Most serial bonds run by years, an equal amount being payable annually. Where the rate is one ending in 5 or 0, and the values for exact interest periods are required, not for intermediate periods, a simpler process may be used, copying values direct from the tables. For example, a series of five 4% bonds of the par value of $1,000 each, J & J, issued July 1, 1914, payable on each first of July, 1915 to 1919, is sold on a 3.50% basis.

Set down in two columns the first ten values from the tables; then add and subtract successively, as follows:

½ yr.	$1002.457	1 yr.	$1004.872	
1½	1007.245	2	1009.577	
2½	1011.870	3	1014.122	
3½	1016.337	4	1018.513	
4½	1020.651	5	1022.753	
			$5069.837	July 1, 1914
Jan. 1, 1915,	$5058.560			
			1022.753	
			$4047.084	July 1, 1915
	1020.651			
Jan. 1, 1916,	$4037.909			
			1018.513	
			$3028.571	July 1, 1916
	1016.337			
Jan. 1, 1917,	$3021.572			
			1014.122	
			$2014.449	July 1, 1917
	1011.870			
Jan. 1, 1918,	$2009.702			
			1009.577	
			$1004.872	July 1, 1918
	1007.245			
Jan. 1, 1919,	$1002.457			

§ 284. Formula for Serials

The total value of an annual series may be obtained by the following formula:

Let m be the number of different maturities and n the number of the periods the last bond has to run. Let r, for brevity, represent the ratio of increase, instead of $1+i$. The powers of r are obtainable from Table I (§ 378), or by logarithms. The principal of each bond being $1, the formula would read:

$$\text{Aggregate premium} = \left(m - \frac{r^{2m}-1}{(r^2-1)r^n}\right) \div i \times (c-i)$$

In the preceding example $m=5$, $n=10$, $r=1.0175$, $i=.0175$, $c=.02$, $c-i=.0025$.

From Table I* or from the "Extended Bond Tables"†:

$$r^2 = 1.03530625$$
$$r^{2m} = r^{10} = 1.18944449$$
$$r^n = r^{10} = 1.18944449$$

Therefore:
$$r^{2m}-1 = .18944449$$
$$r^2-1 = .03530625$$
$$\frac{r^{2m}-1}{(r^2-1)r^{10}} = \frac{.18944449}{.03530625 \times 1.18944449} = 4.511139$$

$$m - \frac{r^{2m}-1}{(r^2-1)r^n} = 5 - 4.511139 = .488861$$

$$\left(m - \frac{r^{2m}-1}{(r^2-1)r^n}\right) \div i = .488861 \div .0175 = 27.93491$$

Value of series $= 5 + (27.93491 \times .0025) = 5.0698373$ which is the result already obtained by addition.

This formula will seldom be of use except in the case of a very complex rate not comprised in the tables. It will then involve the computation of three powers of r by logarithms.

---

* § 378. † Sprague's "Extended Bond Tables."

§ 285. Problems in Valuation of Serial Bonds

The following problems may be solved in either of the ways discussed:

(65) A corporation issued a series of ten $1,000 bonds, 5%, M & N, on May 1, 1913, payable each May 1, 1921 to 1930. What is the value on a 3.60% basis:

(a) On May 1, 1918?
(b) On July 1, 1918?
(c) On August 23, 1918?

(66) Find the values as above, but on a 4% basis.

§ 286. Answers to Problems in Valuation of Serial Bonds

Problem (65)
(a) $10,897.40
(b) $10,962.79 flat.
(c) $11,019.45 flat.

Problem (66)
(a) $10,630.42
(b) $10,701.29 flat.
(c) $10,762.71 flat.

CHAPTER XXVI

OPTION OF REDEMPTION

§ 287. Method of Calculating Income Rate*

The rate of income on a bond subject to a right to redeem at an earlier date than that of actual maturity and on payment of a premium, can be ascertained by means of tables. Only the income which is certain must be calculated upon in advance; hence there will always be a contingent profit which may be realized.

For example, suppose the bond to be a 4½% one absolutely due in 30 years but redeemable at 105 after 20 years; issued 1905, redeemable 1925, payable 1935.

In order to determine where the redemption is a benefit and where it is a disadvantage, we must suppose ourselves to be in 1925 at the redemption date. This bond now has 10 years to run. Turning to the 4½% bond table,† under 10 years, we find that 1.05 is the price almost exactly at a 3.89% basis. Therefore, if the bond is bought now on a 3.89% basis, the investment value in 1925 will be exactly 1.05 and there will be neither profit nor loss in being required to surrender at 1.05; 3.89% may be called the neutral rate.

§ 288. Advantageous Redemption Ignored

It is necessary to bear in mind that the higher the rate of income the lower is the premium; if the rate be more than

* Compare § 147.

† Sprague's "Extended Bond Tables."

3.89%, say 4%, the option may be disregarded, for we shall surely have 4% for 20 years, and probably for the full time. In case the rate of interest has fallen to 3.89%, the issuer of the bond may think it advantageous to redeem, so as to sell his new issue at more than .05 premium. Then, as our bond stands at less than 1.05, we get a profit besides our 4% income. Thus, if the bond is bought at a basis which yields more than 3.89% for 30 years, we may safely amortize at that basis for 20 years, or until the option is exercised.

§ 289. Disadvantageous Redemption Expected

But if the rate for thirty years, which we may call the apparent rate, or non-redemption rate, is less than 3.89%, the bond will be worth more than 105 at the redemption date and the issuer may be expected to redeem. If he does not, it is because the general rate of interest has risen so that he must pay more than 3.89%, in which case he will allow us to continue at 3.89% till maturity. Thus, if the bond is bought at a price which would be on an apparent basis of less than 3.89%, redemption must be expected as being adverse to our interests. The redemption date then becomes the actual date of maturity, but the principal is not 1 but 1.05.

§ 290. Change in Principal

Let the par be \$100,000 and the price \$114,423.38, which is at the apparent basis of 3.70%. To get the actual basis we must consider the par as \$105,000 and the time 20 years. But if the par is \$105,000, the cost is not at 1.1442|| but at 1.1442|| ÷ 1.05 = 1.0897||. The cash rate is also transformed; the cash income is still \$4,500, but this is not 4½% of \$105,000; it is only 4 2/7%.

Therefore, the limitation imposed by the option of redemption entirely changes the problem. Instead of a 4½%

bond for \$100,000, due in 30 years, bought at 1.1442, we have a 4 2/7% bond for \$105,000, due in 20 years, bought at 1.0897.

No tables have been published for 4 2/7% bonds, presumably because this exact case of 4½% bonds redeemable at 1.05 is infrequent. However, we can easily construct them by adding to the value of a 4% bond, 2/7 of the difference between a 4% bond and a 5% bond.

§ 291. Approximate Location

As a rough approximation, find 1.0897 as closely as possible in the 20-year tables for 4% and 5% respectively. The nearest to 1.0897 in the 4% table is 1.08655516, which is a 3.40% income; the nearest in the 5% table is 1.08623676, a 4.35% income. The required rate will be about 2/7 of the distance between 3.40% and 4.35%.

$$\begin{aligned} 4.35 - 3.40 &= .95 \\ 2/7 \text{ of } .95 &= .27 \\ 3.40 + .27 &= 3.67 \end{aligned}$$

Therefore 3.67% is the approximate rate, and we might begin testing with that rate. We notice, however, that the approximations 1.08656|| and 1.08624|| are both short of 1.0897; hence, probably the rate will fall short of 3.67%, and it will be easier to start with the tabular rate 3.65%. In fact, had we gone a little further in decimals, using the colored pages of differences and sub-differences in the bond tables, we should have obtained the following values in the 4% table:

Income rate, 3.38%,	1.08960122
Income rate, 3.37%,	1.09112831

The rate nearest to 1.0897 in the 4% table is therefore 3.38%. Similarly, in the 5% table the nearest rate is

4.33%. Taking 2/7 of the difference between these two rates and adding this difference to 3.38%, gives 3.65% as the approximate income rate.

4% table, 20 years	3.65%	1.0493748
5% " " "	3.65%	1.1904458
Difference		.1410710
1/7		.0201530
2/7		.0403060
Add to 4% value		1.0493748
Giving 4 2/7% value		1.0896808

This value is very close to 1.0897.

Value of $105,000 at the same price..	$114,416.48
Actual price........................	114,423.38
Residue	6.90

This is the nearest approximation we can obtain without using more decimals; therefore, 3.65% is the actual rate of income for a 4½% bond redeemable at 1.05, 10 years before maturity, if purchased at 114.42, 30 years before maturity.

In the diagram (page 244) the dotted line marked 3.70 is the apparent course of a bond at 114.42, 30 years to run; but the option at 105 pulls it down to a 3.65 basis; during the last 10 years it earns 3.89%, if not redeemed. The 4% line, as it passes below the 105 point, is unaffected by the option of redemption. The issuer would not redeem, at 105, a bond whose value was less than 105.

To complete a schedule running from the date of issue to that of redemption, we have the following data:

Par, $105,000.

Cash interest, semi-annually, $2,250, being at the rate of 4 2/7% per annum.

Net income, semi-annually, $1,916.25, being 3.65% per annum on $105,000.

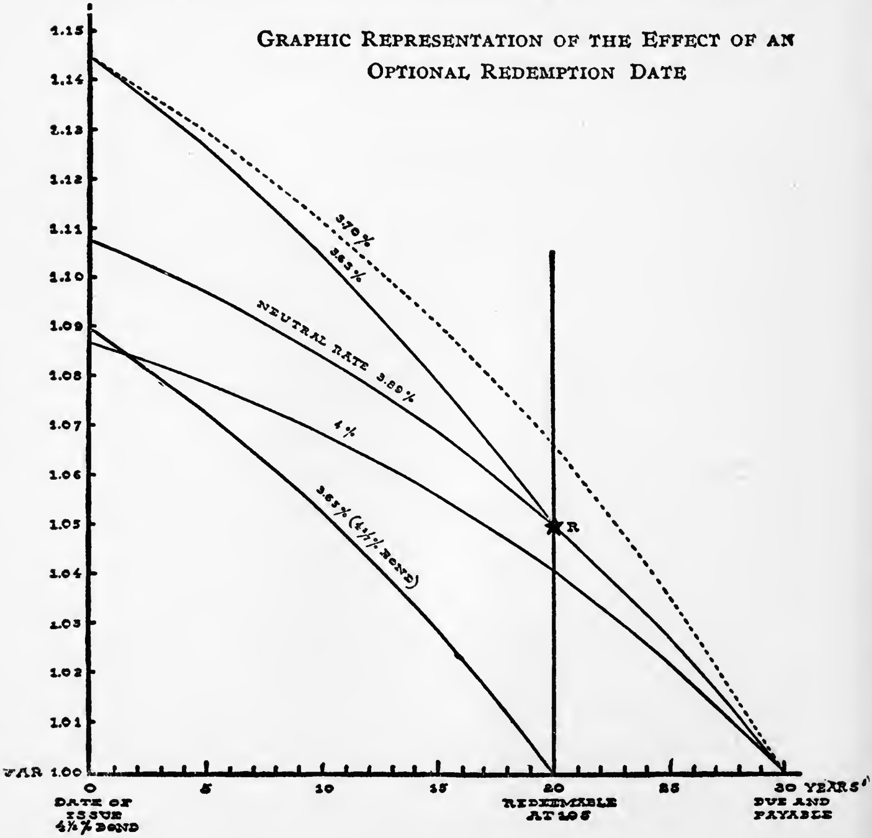
Difference of interest, \$2,250 — \$1,916.25 = \$333.75.

Present worth of 20-year annuity of \$333.75 each half-year, \$9,416.48. Present value of bond at 3.65%, \$114,416.48. Actual value, \$114,423.38. Eliminand, \$6.90.

We might now proceed to amortize \$114,416.48 down to maturity. Each term would then have to be corrected to eliminate the residue, \$6.90. The multiplier for this purpose would be:

$$9423.38 \div 9416.48 = 1.00073276||$$

But we may proceed in the other direction and discount \$333.75 at various dates; this has the great advantage that



GRAPHIC REPRESENTATION OF THE EFFECT OF AN OPTIONAL REDEMPTION DATE

$333.75 may be first multiplied by 1.00073276, thus accomplishing the elimination process once for all.

$$\$333.75 \times 1.00073276 = \$333.99456$$

This last is substituted as a base in place of $333.75, and we proceed to discount, using the factor .982077093||, which is the reciprocal of the semi-annual ratio 1.01825, in the tabular method:

$333.9946	
294.6231	
29.4623	
2.9462	
8839	
884	
39	
6	
$328.0084	½ year before maturity
294.6231	
19.6415	
7.8566	
79	
4	
$322.1295	1 year before maturity
etc.	

§ 292. Problems Involving Optional Redemption Dates

(67) If a 4% bond is redeemable 25 years before maturity at 105, what is the neutral rate of income?

(68) If a bond reads at 4%, but the amount which will be received is 1.05 of the nominal par, what is the actual percentage of cash income?

(69) A 50-year 4% bond is redeemable at 105 after 25

years. Find its actual income rate if bought at (a) 105, (b) 106, (c) 107, (d) 108, and (e) 109.

(70) A 30-year 5% bond is redeemable at 110 after 15 years. Find at what price it should be bought when issued to pay (a) 3.90%, (b) 4.40%.

§ 293. Rule for Determining Net Income

We are now prepared to formulate a rule for determining the net income yielded at a certain price, by a bond bearing a certain par interest but subject to redemption at another price, on the assumption that the right will be exercised.

(1) Divide the nominal cash rate of interest by the redemption price per unit; the quotient will be the actual cash rate, consisting of a whole number and a fraction; e.g., $4\frac{1}{2}\% \div 1.05 = 4\ 2/7\%$.

(2) Divide the purchase price by the same divisor, giving the actual purchase price per unit; e.g., $1.1442 \div 1.05 = 1.0897$.

(3) Select two different bond tables, one at a lower, one at a higher cash rate than the actual rate obtained in paragraph 1. These should be even rates, not fractional, and 1% apart. Find the column for the number of years before redemption; e.g., 4% and 5%, 20 years.

(4) In each of these columns find the nearest price to the actual purchase price in paragraph 2; e.g., in 4% table, 1.08656; in 5% table, 1.08624.

(5) Set down the two rates of net income found opposite these values, and find their difference; e.g., inter-rates, 4.35% and 3.40%; difference, .95%.

(6) Take such a fraction of the difference as is shown by the fractional part of the mixed number which represents the actual cash rate; add the result to the smaller rate and

the sum is, approximately, the desired yield; e.g., 2/7 × .95 = .27; .27 + 3.40 = 3.67.

(7) Try the nearest rates from the table until one is found which produces the desired price; e.g., 3.65 produces 1.08968.

§ 294. Answers to Problems Involving Optional Redemption Dates

Problem (67)
3.69% +

Problem (68)
3 17/21%, or 3.80952% +

Problem (69)
(a) Between 3.77% and 3.78%; (b) 3.73% +; (c) 3.69%; (d) 3.63%; (e) 3.57% +

Problem (70)
(a) 118.005676; (b) 109.042757

CHAPTER XXVII

BONDS AT ANNUAL AND OTHER RATES

§ 295. Standard of Interest

In popular usage and, in fact, in legalized usage, though not from the mathematical standpoint, the interest on a given principal is directly proportional to the time; that is, if the interest is six dollars on a hundred for a year, it must for six months be three dollars, and for three months one dollar and a half. These three rates are popularly regarded as identical, but actually they are very different. A single standard should be preserved, and when in any problem "6 per cent" is once taken as meaning "3 per cent per half-year," it must not be arbitrarily shifted to mean "1½ per cent per quarter," which is really "3.0225 per cent per half-year."

If the ratio of increase or income yield be kept at the same unvarying standard, the frequency of collection, or cash payment, affects the value of the investment. To change the coupon from half-yearly to quarterly, must necessarily enhance the value of the annuity made up of the coupons. The nearer any one of them approaches to the present, or the less time one must wait for his money, the more nearly is it worth its par; while the present worth of the principal remains the same, unless we vary the income yield.

§ 296. Semi-Annual and Quarterly Coupons

A bond for $1,000,000, due in one year, bearing semi-annual coupons at 6 per cent per annum, at a price to net

2½ per cent computed semi-annually (1¼% per period), is worth, according to all tables and computations (except the fictitious one of "reinvestment" at an arbitrary rate) $1,034,354.52
thus,

Present worth of first coupon, $30,000, one period, 1¼%	$ 29,629.63
Present worth of second coupon, $30,000, two periods, 1¼%	29,263.83
Present worth of principal, $1,000,000, two periods, 1¼%	975,461.06
By the method in § 111...........	$1,034,354.52

Or, using the method in § 115, we should take the

nominal interest	$30,000
subtract from it the effective interest.........	12,500
and obtain the interest-difference..........	$17,500
An annuity of $17,500, for two terms, at 1¼%, would be the premium...........	$34,354.52

The company issuing the bonds is willing in return for certain concessions to make its interest payments quarterly. How much would this add to the value of the bond, the income yield being still 2½ per cent on a semi-annual basis?

If the bond be made quarterly, the same cash is received each half-year, but $15,000 of it is received three months earlier than before. On this $15,000 the bondholder is entitled to only 3 months' interest, instead of 6 months, at 1¼% per half-year; therefore, a quarter's interest on this quarterly coupon must be deducted each half-year from the entire interest earned.

We must be careful, however, to compute the interest correctly on this advanced coupon. It must be at .00623059,

not at .00625. Interest at a half-period is not half of the .0125, but the square root of the ratio 1.0125, less the 1; $\sqrt{1.0125} = 1.0062305911$, interest $= .00623059$. Otherwise we should be using a higher rate than 1.0125 for the half-year, nearly 1.01254. The interest to be deducted each half-year is \$15,000 × .00623059 = \$93.46. The effective interest is \$12,500 — \$93.46 = \$12,406.54, and the interest-difference \$30,000 — \$12,406.54 = \$17,593.46. If we should now consider each instalment of the annuity to be \$17,593.46 instead of \$17,500, we should have the premium for quarterly coupons. Therefore, the two annuities (or, in other words, the premiums) at any point must be to each other as \$17,593.46 : \$17,500; or the ratio of the quarterly premium to the semi-annual is 1.005340507. Hence the multiplier .0053405 on page VII of Sprague's "Extended Bond Tables."

In symbols, the income rate becomes (instead of i), $i - \frac{c}{2}(\sqrt{1+i}-1)$ and the interest-difference becomes (instead of $c - i$), $c - [i - \frac{c}{2}(\sqrt{1+i}-1)] = c - i + \frac{c}{2}(\sqrt{1+i}-1)$, which divided by $(c - i)$ gives the proportion $1 + \dfrac{\frac{c}{2}(\sqrt{1+i}-1)}{c-i}$ = (in the above case) 1.0053405.

The process of finding .0053405 may be briefly expressed thus:

Rule: Divide a quarter's interest on a quarterly coupon by the interest-difference.

The value of the bond when trimestralized (reduced to a quarterly basis) is, therefore:

At semi-annual payments	\$1,034,354.52
Added for quarterly coupon, 34,354.52 × .0053405	183.47
Value trimestralized	\$1,034,537.99

This may be tested by multiplying down to maturity, 3 months at a time, viz.:

\$1,034,537.99 × .00623059	+ 6,445.79
	\$1,040,983.78
	− 15,000.00
	\$1,025,983.78
\$1,025,983.78 × .00623059	+ 6,392.48
	\$1,032,376.26
	− 15,000.00
	\$1,017,376.26
\$1,017,376.26 × .00623059	+ 6,338.86
	\$1,023,715.12
	− 15,000.00
	\$1,008,715.12
\$1,008,715.12 × .00623059	+ 6,284.88
	\$1,015,000.00
Final payment,	1,015,000.00

We will now take an example where the effective rate is greater than the cash rate. A bond of \$1000 at 4% (sem.), due in ten years, is bought so as to give a net income of 5% (sem.); what will be its value if trimestralized?

The normal or semi-annual value is by all tables..\$922.054

The discount, $\frac{i-c}{i} \times \left(1 - \frac{1}{(1+i)^n}\right)$, is..... 77.946

The multiplier* is..................................0248457

77.946 × .0248457, amount of added value, = ...\$1.93661

922.054 + 1.937 = ... 923.991

* Sprague's "Extended Bond Tables," page VII.

§ 297. Shifting of Income Basis

This is the correct value, the income basis being unchanged. But in some recent books we find the quarterly value to be stated as $921.683, which is a surprising result, for we should not expect the value of the security to be *diminished* by a more frequent interest-payment. The trouble is, that the income basis has been suddenly shifted from 4% semi-annual to 4% quarterly, and we are given comparisons between the following values:

(a) At 4% semi-annual basis, coupon 5% semi-annual.

(b) At 4% quarterly basis, coupon 5% quarterly.

Whereas the value really desired is:

(c) At 4% semi-annual basis, coupon 5% quarterly.

In all the tables using the basis (b), the values below par are all apparently diminished by frequency of payment. The author's tables are computed on the semi-annual income basis, though the coupons may be quarterly or annual.

§ 298. Problems—Bonds at Varying Rates

(71) A 5% quarterly bond for $100,000 has 5 years to run on a 4% semi-annual basis; what is its value?

(72) Ascertain the value of the same bond at 4½ years.

(73) Derive the 4½ years' value from the 5 years, and obtain the same value as in (72).

(74) Find the value of a 2% quarterly bond, 5 years to run, which nets 1.80% semi-annually.

(75) Two issues of 20 year, 3½% bonds, each $100,000, are offered; one with interest semi-annually at 95.29, the other quarterly at 95.38; find the better purchase.

(76) Which is the better purchase:

$1,000,000 4% quarterly bonds, 10 years, at 104.33, or

$1,000,000 3% semi-annual bonds, 10 years, at 95.50?

§ 299. Answers to Problems—Bonds at Varying Rates

Problem (71)

$104,603.02

Problem (72)

$104,182.64

Problem (73)

Value at 5 years	$104,603.02
Of this $1,250 is payable in three months.	
Present worth at 4% semi-annually	1,237.69
The remainder	$103,365.33
Produces income at .02	2,067.31
	$105,432.64
Cash interest received	1,250.00
Value at 4½ years	$104,182.64

An alternative solution for this problem, and the one usually employed, is as follows:

Value at 5 years	$104,603.02
This multiplied by the quarterly effective rate, .00995049 (which is the square root of 1.02, less 1) gives	1,040.85
	$105,643.87
Less quarterly coupon	1,250.00
Giving value at 4¾ years	$104,393.87
This multiplied by .00995049 gives the next quarterly income	1,038.77
	$105,432.64
Less quarterly coupon	1,250.00
Giving value at 4½ years	$104,182.64

Problem (74)

$100,973.61

Problem (75)

The quarterly bonds.

Problem (76)

The semi-annual bonds.

§ 300. Bonds with Annual Interest—Semi-Annual Basis

Bonds on which the interest is paid only once a year are somewhat rarer than those where it is paid four times a year; but, when they do occur, means should be provided for ascertaining their value at any given rate reduced to the standard of semi-annual income. This is somewhat easier than finding the value of a quarterly bond on a semi-annual income basis.

We may begin by a simple example using the discount method, either by division or by multiplication, taking a 4% annual bond yielding 3% semi-annually, 2 years to run, for $100,000.

Beginning at maturity at par............	$100,000.00
and adding to it the annual coupon then due..	4,000.00
	$104,000.00
We discount this by dividing by the ratio, 1.015, or, what is the same thing, multiplying by its reciprocal, .98522167; $104,000 ÷ 1.015 or × .98522167 =...............	$102,463.05
This is the value, flat, 6 months before maturity. If there were a payment of interest at this date we should add its value. But there is none; hence we continue the process, $102,463.05 ÷ 1.015 or × .98522167 =.......	$100,948.82

Here we add the coupon payable one year before maturity	4,000.00
	$104,948.82
We discount this for another half-year, $104,-948.82 ÷ 1.015	$103,397.85
and again, $103,397.85 ÷ 1.015 . . . . which is the value required.	$101.869.81

To test this, let us multiply down to maturity:

Value at 2 years	$101,869.81
Income at 1½%, ½ year	1,018.70
	509.35
Value at 1½ years, flat	$103,397.86
No coupon.	
Income at 1½%, ½ year	1,033.98
	516.99
	$104,948.83
Annual coupon paid	4,000.00
Value at 1 year	$100,948.83
Add ½ year's income, at 1½%	1,009.48
	504.74
	$102,463.05
Add last ½ year's income, at 1½%	1,024.63
	512.32
Total principal and interest	$104,000.00

§ 301. Annualization

We will now annualize the above process; that is, instead of multiplying twice by 1.015, we will multiply once by 1.030225, which is 1.015 × 1.015, or $(1.015)^2$.

As before, beginning with..................	$101,869.81
we multiply by 1.030225....................	3,056.09
	20.37
	2.04
	.51
	$104,948.82
and subtract the coupon....................	4,000.00
	$100,948.82
again multiply by 1.030225.................	3,028.47
	20.19
	2.02
	.50
giving the same result.....................	$104,000.00

Thus, income has been received on all of the investment outstanding at 1.5% per half-year, or at 3.0225% per year.

§ 302. Semi-Annual Income Annualized

Suppose now that we take the case of an ordinary half-yearly bond paying a cash interest of 2% twice a year, and yielding 1.5% half-yearly, with the purpose of annualizing in this case also. The ratio, when annualized, is the same as before, 1.030225, but there are two semi-annual coupons of $2,000.00 each, instead of the single annual coupon of $4,000.00 as in the previous case. The first of these coupons, if deferred to the end of the year, will increase at

the semi-annual ratio of 1.015 to..............	$2,030.00
The second coupon remains......................	2,000.00
The entire cash interest, when concentrated at the end of the year, is therefore equivalent to....	$4,030.00

The processes of multiplying down to maturity, using both semi-annual and annual periods, are shown below side by side, beginning with the value $101,927.19 found from

tables or by computation. In a third column appears the annualized process in the case of a 4% annual coupon. In all three cases, the net income is 1.5% per half-year, or its equivalent, 3.0225% annually.

Cash Interest 2% per Half-Year		Cash Interest 4% per Year
Ordinary Process	Annualized Process	Annualized Process
$101,927.19	$101,927.19	$101,869.81
1,019.27	3,057.82	3,056.09
509.64	20.38	20.37
	2.04	2.04
$103,456.10	.51	.51
2,000.00		
$101,456.10		
1,014.56		
507.28		
$102,977.94	$105,007.94	$104,948.82
2,000.00	4,030.00	4,000.00
$100,977.94	$100,977.94	$100,948.82
1,009.78	3,029.34	3,028.47
504.89	20.20	20.19
	2.02	2.02
$102,492.61	.50	.50
2,000.00		
$100,492.61		
1,004.93		
502.46		
$102,000.00	$104,030.00	$104,000.00

§ 303. Comparison of Annual and Semi-Annual Bonds

In each of these columns the proper principal is attained at maturity, together with its accompanying interest, either actual or annualized. Observing the first and second columns, we see that a semi-annual 4% bond is effectively a 4.03% annual bond, the net income in both cases being 3.0225% per annum. Comparing the second and third columns, the point to be noted is that their chief difference lies in the effective cash rates, one being 4.03% and the other 4%; in the semi-annual bond, annualized, the interest-difference between the cash and income rates is

$4,030.00 — $3,022.50 = $1,007.50

In the annual bond, it is.. 4,000.00 — 3,022.50 = 977.50

§ 304. Finding Present Worth of an Annuity

These interest-differences, $1,007.50 and $977.50, are important because (according to the second rule in Chapter X) we have only to multiply these two interest-differences by the present worth of an annuity of $1 for 2 periods at 3.0225%, in order to obtain the respective bond premiums. We might find this present worth approximately from Table IV* by interpolation between the 3% and 3½% columns, but a much more accurate result may be obtained by the use of Table II*, where we can find the present worth of $1 for 4 periods at 1.015, which is exactly equivalent to the present worth of $1 for 2 periods at 1.030225.

This value........................$ .94218423
must, according to Chapter V, be subtracted
from.............................. 1.00000000

and the remainder.....................$ .05781577
must be divided by the income rate........... .030225
The quotient is...........................$1.9128460

* In Chapter XXXII.

which is the present value of an annuity of \$1 for 2 periods at 3.0225% per period. The foregoing is an application of the two symbolic rules, $D = 1 - p$ and $P = D \div i$.

In order to obtain the bond premiums, we must multiply the above present worth by 1,007.50 in the case of the semi-annual bond, and for the annual bond by 977.50.

Premium on semi-annual bond .... \$1.912846 × 1,007.50 = \$1,927.192

Premium on annual bond 1.912846 × 977.50 = 1,869.807

These premiums agree perfectly with the values previously obtained otherwise, viz.: \$101,927.19 and \$101,869.81.

As another example, take that of a 4% annual bond yielding 5%, for two years. Evidently this will be at a discount instead of at a premium. To annualize the ratio 1.025, multiply it by itself, giving 1.050625; the annualized

interest rate is therefore........................	.050625
from this subtract...........................	.04
giving as the interest-difference................	.010625

To find the present worth of an annuity of \$1 for 2 (annual) periods at 5.0625%, take from Table II,* column 2½%, the value for 4 (semi-annual) periods..\$ .90595064

subtract from	1.00000000
The compound discount is therefore..........	\$ .09404936
Divide by .050625; the quotient is...........	\$1.8577652

which is the required present worth of an annuity of \$1 for 2 periods at 5.0625%.

Multiply this by .010625; \$1.8577652 ×

.010625 = ...	\$ .01973876
This is the discount, which, subtracted from par	1.00000000
gives the value of a \$1 bond.................	\$ .98026124

* In Chapter XXXII.

This may be tested by multiplying down to maturity:

$\times$ 1.025	.02450653
	$1.00476777
$\times$ 1.025	.02511919
	$1.02988696
	.04
	$ .98988696
$\times$ 1.025	.02474718
	$1.01463414
$\times$ 1.025	.02536586
	$1.04000000
	.04
	$1.00

§ 305. Rule for Bond Valuation

We are now prepared to formulate a rule for valuing an annual bond on a semi-annual basis without reference to the values of a corresponding ordinary (or semi-annual) bond.

Rule 1:

(a) Annualize the rate of interest (find the equivalent annual income rate); e.g., $1.015^2 = 1.030225$.

(b) Subtract this rate from the annual coupon, or *vice versa,* to give the interest-difference; e.g., $.04 - .030225 = .009775$.

(c) Multiply the latter by the present worth of an annuity of $1 for the number of annual periods at the annualized rate, giving the premium or the discount; e.g., $.009775 \times 1.9128453 = .0186981$.

Where the values of the ordinary semi-annual bond

have already been calculated, as in the bond tables, it will be possible to obtain therefrom the values of the annual bond, with a saving of time.

§ 306. Multipliers for Annualizing

For each combination of a cash rate with an income rate, a multiplier may be found which, applied to the premium or the discount for any number of years on a semi-annual bond, will give the depreciation caused by the collection of the interest once a year only; and this multiplier will be constant, whatever the time. A table of these multipliers will be found in Sprague's "Extended Bond Tables," page VIII.

In the example given in § 300 we have a 4% annual bond yielding 3% semi-annually. On page VIII* in the column headed "4% Bond" on the line opposite "3%" is the multiplier .0297767. The premium on the ordinary semi-annual bond for $100,000 at 2 years, we have seen, is $1,927.19.

$1,927.19 × .0297767 = ...$	57.385
As the value, if semi-annual, would be...	101,927.192
the value of the annual bond is reduced to	101,869.807

In the example in § 304, the annualizer, or multiplier, for a 4% bond to yield 5% is found from the table

to be......................................	.0493827
The value of a semi-annual bond of $1 at 2 years is................................	$.98119013
or its discount is........................	.01880987
$.01880987 × .0493827 = ...	.00092888
which subtracted from.....................	.98119013
gives the annualized value................	.98026125
This differs from the one already given....	.98026124

by 1 cent on a million dollars, owing to decimals having been rounded off.

* Sprague's "Extended Bond Tables."

These multipliers are obtained by the following formula, in which c and i represent the nominal rates per annum.

$$\frac{ci}{(4+i)\ (c-i)}$$

§ 307. Formula for Annualizer

The formula may be thus expressed as a rule.

Rule 2: To find the annualizer for any two rates:

(a) Multiply the rates together for a dividend; e.g., $.04 \times .03 = .0012$.

(b) Multiply 4 + the income rate, by the difference of rates for a divisor; e.g., $4.03 \times .01 = .0403$.

(c) Their quotient will be the required multiplier, or annualizer; e.g., $.0012 \div .0403 = .029776675$.

The product of the premium by the annualizer is always subtracted from the semi-annual value; and sometimes the resulting value may be shifted to a discount from a premium, even if it was a premium which was extracted from the table. Thus, in the case of a $1,000,000 5% annual bond, payable in one year, netting 4.95%, the premium $482.03 × the annualizer 1.22237313 = $589.22, and the value of the annual bond becomes $999,892.81.

It must be observed that only values for full years can be obtained in either of these ways. An odd half-year is a "broken" period, and must be treated as in Chapter XI.

§ 308. Conventional Process

While the foregoing is the method which would doubtless be followed in buying and selling, a more accurate result, from a mathematical standpoint, would be obtained by using as the half-year value the one found by multiplying down at the effective rate.

Thus, in a bond at 4%, payable annually, on a 3% semi-annual basis, the values are:

2 years before maturity...	$1,018,698.07
1 year " " ...	1,009,488.22
Maturity	1,000,000.00

The amortization for the first year is $9,209.85, and for the second $9,488.22. Halving these severally, the values by half-years appear as follows:

	Values	D_1
2 years	$1,018,698.07	$4,604.92
1½ years	1,014,093.15	4,604.93
1 year	1,009,488.22	4,744.11
½ year	1,004,744.11	4,744.11
Maturity	1,000,000.00	

§ 309. **Scientific Process**

The foregoing result would be in accordance with the conventionally established rule that during any period (which is here a year) simple interest must prevail and the amortization accrue proportionately to the time elapsed from the beginning of the period.

But the half-year may, with equal propriety, be considered the period, since the income is on a semi-annual basis. Under this assumption we must multiply down:

Value, 2 years............	$1,018,698.07	
× 1.015.....................	10,186.98	
	5,093.49	
Value, 1½ years...........	$1,033,978.54	flat
Less accrued interest.......	20,000.00	
Value, 1½ years...........	$1,013,978.54	and interest

Similarly, the value at one-half year is fixed at $1,004,-630.54, and the series with differences will appear as follows:

	Values	D_1
2 years	$1,018,698.07	$4,719.53
1½ years	1,013,978.54	4,490.32
1 year	1,009,488.22	4,857.68
½ year	1,004,630.54	4,630.54
Maturity	1,000,000.00	

In the second half of each year there is less amortization, and consequently more earning than in the first half; but this may be defended on the ground that by the conditions prescribed, interest is compounded semi-annually. The earning power at compound interest must continue to increase until a cash payment; and there is no cash payment at the mid-year.

§ 310. Values Derived from Tables

This latter form of valuation at mid-years is recommended for comparative (non-commercial) purposes.

The values at $n\frac{1}{2}$ years, $1,004,630.54, $1,013,978.54, etc., may be deduced from the ordinary extended tables by multiplying by the annualizer, with this proviso: that the interest-difference must first be temporarily added to the tabular premium or discount before multiplying. Thus, in the case just considered, the excess of .02 over .015 is .005 each half-year; or, on $1,000,000, $5,000.

To find the value for 1½ years, take from the

table the premium..........................	$14,561.00
add the interest-difference......................	5,000.00
giving the multiplicand........................	$19,561.00
which, multiplied by the annualizer .9702233, equals..................................	$18,978.54
from which again subtract....................	5,000.00
giving the premium as above..................	$13,978.54

§ 311. Successive Process

In general, when a schedule is to be formed for an annual or a quarterly bond, on a semi-annual basis, it will be found easier after ascertaining the initial value to multiply down to maturity, as that will usually require fewer figures.

§ 312. Problems and Answers—Varying Time Basis

(77) $25,000 4% bonds, interest payable annually, 8 years to run; what is the price at a 3.70% semi-annual basis?

(78) What multiplier will annualize the premium on the above bonds as given in the regular bond table?

(79) An offering is made of $30,000 3½% bonds, interest payable annually, of which $10,000 mature in one year and $10,000 each year thereafter. What should be paid for them to produce 3.40% semi-annually?

Answers:

Problem (77)
$25,452.30

Problem (78)
.87779704||

Problem (79)
$30,040.34

§ 313. Bonds at Two Successive Rates

Occasionally bonds are issued with the agreement that the interest paid shall be at a certain rate for some years, and at another rate for the remainder of the time to maturity. An example is a fifty-year bond bearing 4% for 20 years and 5% for the following 30 years. The problem is then to find the price at which they will pay a certain income, say 3.60%.

Each of the two successive cash rates will cause a premium, and we may calculate these premiums separately.

§ 314. Calculation of Immediate Premium

The premium caused by the 4% rate will last only 20 years and will then vanish; hence, this premium is just the same as that on a plain 4% bond for 20 years, netting 3.60%, which we find by calculation or from tables to be $56,680.10 on $1,000,000.

§ 315. Calculation of Deferred Premium

The premium produced by the 5% rate does not take effect immediately, but after 20 years. It is a deferred annuity. An annuity for the entire 50 years of the excess interest, 1.40%, or in other words the premium on a fifty year 5% bond to net 3.60%, is.............. $323,568.65

But during the first 20 years there will be no such premium; we have already charged that at 4%. Hence we must by subtraction eliminate the analogous 5% premium for 20 years, which is 198,380.36

leaving a remainder......................... $125,188.29

which is the premium, or present worth, of the enjoyment of a 5% cash rate (as against a 3.60% income rate) commencing 20 years from date and continuing till 50 years from date.

Adding together the two premiums, $56,680.10 and $125,188.29, we have $181,868.39 as the premium which should be paid for the bond.

A simpler way to apply the principle is to add together

the 4% value for 20 years.................	$1,056,680.10
and the 5% value for 50 years..............	1,323,568.65
	$2,380,248.75
and subtract the 5% value for 20 years......	1,198,380.36
giving the value of the composite bond......	$1,181,868.39

This procedure has the advantage that it applies alike to bonds which are selling at a premium and to those which are selling at a discount and automatically allows for that distinction.

§ 316. Symbols and Rule

We may for convenience represent the earlier rate by c_1 and the latter rate by c_2, i being the net income. We may put m for the number of years at which the rate c_1 prevails, and n for the number of years at c_2; $m + n$ is the entire time. The rule will then be as follows:

Rule: To find the value of a bond to yield i per cent, when by its terms it pays cash interest at the rate c_1 for m years and thereafter at c_2 for n years, maturing in $m + n$ years. Add together the value of a c_1 bond for m years and that of a c_2 bond for $m + n$ years, and from the sum subtract the value of a c_2 bond for m years.

An example of a bond of very early maturity will illustrate the principle of the rule and will admit of demonstration by multiplying down. A bond for $100,000 paying 5% for 1 year (2 periods) and 6% thereafter for 1½ years (3 periods) is to be valued so as to yield the annual return of 4%.

§ 317. Analysis of Premiums

If the rate on the bond were 5% for the entire 2½ years, its value, according to the bond tables,* would be $102,356.73. On the other hand, if the rate were 6%, its value would be $104,713.46. Let us analyze these premiums into their component parts, which are the present worths of excess interest for five periods, $500 per period in the case of the 5% bond, and $1,000 per period in the case of the 6% bond.

* Sprague's "Extended Bond Tables."

	5%	6%
½ year	$490.196	$980.392
1 year	480.584	961.168
Premium one year before maturity	$970.780	$1,941.560
1½ years	471.161	942.322
2 years	461.923	923.846
2½ years	452.866	905.732
Premium 2½ years before maturity	$2,356.730	$4,713.460

Any premium is the sum of a certain number of present worths of $500 or of $1,000. But in the double-rate bond, the only present worths that have an influence on the inaugural value are the first two in the 5% column and the last three in the 6% column, as indicated by the braces placed opposite them.

It is evident that the values producing premiums at the 5% rate amount to $970.78, and that those in the 6% column amount to $2,771.90 (the easiest way to obtain this latter amount being to subtract $1,941.56 from $4,713.46). Hence the premium is:

$$\$970.78 + \$2{,}771.90 = \$3{,}742.68$$

The equivalent process by the rule would be:

Value of c_1 bond, m years...	$100,970.78
plus Value of c_2 bond, $m+n$ years	104,713.46
	$205,684.24
less Value of c_2 bond, m years...	101,941.56
	$103,742.68

It will be interesting to multiply down to maturity and thus test this result:

	$103,742.68
+	2,074.85
	$105,817.53
−	2,500.00
	$103,317.53
+	2,066.35
	$105,383.88
−	2,500.00
	$102,883.88
+	2,057.68
	$104,941.56
−	3,000.00
	$101,941.56
+	2,038.83
	$103,980.39
−	3,000.00
	$100,980.39
+	2,019.61
	$103,000.00
−	3,000.00
Par	$100,000.00

It will sometimes be the case that in multiplying down the values will increase for a time and then begin to decrease at the change of rate; or *vice versa,* the values will at first decrease and then later increase.

§ 318. Problems and Answers—Successive Rates

(80) An issue of bonds matures on Jan. 1, 1966. Interest is to be at 5% till Jan. 1, 1936, and thereafter at 6%. What is the price at a 3.60% basis on July 1, 1916?

(81) $10,000 of Waterworks Bonds, 5 years to run, first 3 years at 4%, thereafter at 5%; find the value to yield 4.40%.

(82) Find the value of the same bonds to net 4¾%; 5¼%.

Answers:

Problem (80)
$1,413,422.66

Problem (81)
$9,988.49

Problem (82)
$9,824.98; $9,617.04

CHAPTER XXVIII

REPAYMENT AND REINVESTMENT

§ 319. Aspects of Periodic Payment

When a loan is payable in equal periodic instalments, each covering the interest and part of the principal, the most obvious way of looking at it is that the principal is gradually paid off; and then we have this aspect:

(1) A diminishing principal;
A diminishing interest charge, and therefore
An increasing repayment.

But precisely the same result may be obtained from a different point of view by assuming that no payment is made at all until the final date of maturity, at which time the sinking fund, or sum of instalments plus interest, is just sufficient to pay off the whole debt. In this case, we will have the following aspect:

(2) An unchanged principal;
A uniform interest charge;
A uniform instalment devoted to reinvestment and allowed to accumulate.

As an illustration of the first aspect, suppose we consider a debt of $1,000, bearing interest at 3% per period. This debt may be extinguished in four periods by uniform instalments of $269.03 at the end of each period, as we have already pointed out in Chapter VII. For convenience, however, we again set forth the details on the following page:

Instalment		Interest on Balance	Payment on Principal	Principal Outstanding
				$1,000.00
(1)	$269.03	$30.00	$239.03	760.97
(2)	269.03	22.83	246.20	514.77
(3)	269.03	15.44	253.59	261.18
(4)	269.03	7.85	261.18	0.
Total,	$1,076.12	$76.12	$1,000.00	

Here we see the diminishing principal, the diminishing interest charge and the increasing repayment or amortization.

From the reinvestment point of view, we have:

Instalment		Interest on Entire Principal	Carried to Sinking Fund	Principal
				$1,000.00
(1)	$269.03	$30.00	$239.03	1,000.00
(2)	269.03	30.00	239.03	1,000.00
(3)	269.03	30.00	239.03	1,000.00
(4)	269.03	30.00	239.03	1,000.00

For Reinvestment		Interest on Previous Total	Total Accumulated
(1)	$239.03		$ 239.03
(2)	239.03	$ 7.17	485.23
(3)	239.03	14.56	738.82
(4)	239.03	22.15	1,000.00

The amortization of principal in its two aspects as repayment and reinvestment should be carefully studied and

the problems in connection with Chapter VII should be worked over into schedule form in each aspect.

§ 320. Integration of Original Debt

This principle will be found to hold: The "principal outstanding" by the first method + the "total accumulated" by the second method = the original debt.

The first point of view is based entirely on facts. Without regard to reinvestment, it is certain that the borrower pays and the lender receives the exact rate of interest stipulated for each period on the actual balance due at the beginning of such period, and this balance may be represented either by a single account or by a cost account and an annulling account.

§ 321. Use of the Reinvestment Point of View

There are some cases where, especially from the point of view of the debtor, it is desirable to keep in view the entire original sum. One of these cases is where it is impossible or impracticable to diminish or pay off the debt before maturity and where accumulation is the only method available. Another is that of a trust where there is an obligation to keep the *corpus* of the fund intact, and consequently reinvestment in some form is necessary.

But the calculations of reinvestment are hypothetical and prospective. They have not the same actuality as those of repayment, but are theoretical estimates of what is expected. Unless a contract has been made to take the instalments off one's hands at a fixed rate, the amount realized is pretty sure to differ from the amount anticipated.

§ 322. Replacement

There is a third method of considering periodic payments, which is not mentioned in the actuarial treatises, and

which may be called replacement to distinguish it from repayment and reinvestment. The successive repayments are transferred to new investments, which are not to accumulate but merely to furnish new income, helping out the diminished income on the waning principal. We have outlined this procedure under "Bonds as Trust Fund Investments," in § 148; but for purposes of comparison we will put the materials already used in § 319 into the replacement form, assuming at first that replacements are so invested as to earn exactly 3%.

	1 Interest on Principal	2 Payment on Principal	3 Principal Unpaid	4 Replacement	5 Interest on Replacements	6 Total Income 1+5
			$1,000.00			
(1)	$30.00	$ 239.03	760.97	$ 239.03		$ 30.00
(2)	22.83	246.20	514.77	246.20	$ 7.17	30.00
(3)	15.44	253.59	261.18	253.59	14.56	30.00
(4)	7.85	261.18	0.	261.18	22.15	30.00
	$76.12	$1,000.00		$1,000.00	$43.88	$120.00

Column 4 of replacements is not accumulative, as its interest is not compounded, but is used as income, supplementing that in Column 1. The balance of Column 3 plus the total of Column 4 at any point make up $1,000. The two corresponding amounts in Columns 1 and 5 always make up $30 (Column 6). At the close, the original $1,000 has been exactly replaced by the new securities.

§ 323. Diminishing Interest Rates

As already remarked, it would seldom happen that exactly 3% would be the rate secured for the replacements, which ought to be of the same grade of security and availability as the original sum. Let us suppose that the rate of interest was declining so that the first replacement had to

be loaned at 2.95%, the second at 2.90%, and the third at 2.75%. Columns 5 and 6 are then the only ones changed:

5 Interest on Replacements	6 Total Income 1+5
	$ 30.00
$ 7.05	29.88
14.19	29.63
21.16	29.01
$42.40	$118.52

Here we have the principal intact, and the falling-off is a gradual one affecting the interest. If we had proceeded on plan No. (2), the full predicted interest would have been consumed, but the principal would have been impaired, which is inadmissible. Hence, in cases of this kind, we must use the vanishing principal with actual replacement. The reinvestment scheme is a basis of calculation only and cannot, like the repayment plan, be reduced to practice.

§ 324. Proof of Accuracy

It is interesting to note that in the repayment method the work may at any point be tested by a fresh calculation, showing the whole procedure to be coherent and consistent. For instance, in our example, the principal at three periods from maturity is $760.97. Treating this as the principal, to find the sinking fund we divide $760.97 by 2.82861, just as for 4 periods we divided $1,000 by 3.7171. This gives $269.03—the same result as before for the value of the equivalent annuity; $22.83 as the interest ($760.97 × .03), and $246.20 as the first repayment or the constant reinvestment, in either aspect.

§ 325. Varying Rates of Interest

It must not be supposed that there is at any one moment a single rate of interest prevailing. Considerations of security, convenience, and availability give rise to different grades of securities and different rates of interest. The prudent investor will probably have at the same time some capital out at high rates and some at low. The money at high rates is not quite so secure, not quite so readily realizable, and requires more effort for the collection of its income. That at low rates is nearer to absolute freedom from risk and from the labor of supervision; it almost automatically collects its own income. The investor will have so planned his investments as to endeavor to preserve a judicious equilibrium between different grades of security, and consequently of income. As his investments are liquidated, he will try to maintain or improve this equilibrium, and he will choose his reinvestments from a wide range, some of low revenue but highest safety and others of the contrary qualities. It is therefore fallacious to assume that, as an author has said, "on the same day and under the same circumstances money received from any one source may be invested at the same rate as that received from any other source." Theoretically it may be, but practically it will usually be invested in the same grade of security as that which it replaces.

§ 326. Dual Rate for Income and Accumulation

When the lender assumes great risk, or when the supply of loanable capital is temporarily deficient, he will exact very high rates, or refuse to loan. Or he may require a high rate and also demand that the instalments of repayment shall be large enough to secure the higher rate on the entire original loan until fully paid; while in ordinary reinvestments a lower rate is easily obtained.

§ 327. Instalment at Two Rates

Suppose that \$1,000 is loaned, repayable in 4 instalments, on such a basis that the lender will have 5% interest per period on the entire capital, while it will be replaced by accumulating at 3%.

The sinking fund is exactly the same as in our previous example, \$239.03. But the instalment is:

not \$30 + \$239.03, or \$269.03
but \$50 + \$239.03, or \$289.03

The instalment here is as much greater as the interest is greater. The accumulation is precisely the same as heretofore. The instalment provides not only 5% on the money remaining invested, but also 2% (unearned) on that which had been repaid.

An instalment of only \$282.01 would pay 5% on the outstanding capital, which would gradually be replaced by 3% investments. Thus it is seen that the borrower has to pay more than 5%; in this instance about 6.066%.

§ 328. Amortization of Premiums at Dual Rate

This loaning at a dual rate is of so little practical importance, at least in this country, that it would not be worth mentioning, except that a few writers have tried to apply the same principle to the amortization of premiums. They assume that there is no other way of ascertaining the value of a bond than by laying aside the excess of interest and letting it accumulate till maturity. But this is not at all necessary. The question is, what uniform rate is yielded by each dollar of the investment during the time it is outstanding. When this is ascertained, it can make no difference what is done with the capital after it is returned. We may as well say that the rate of a series of bonds payable \$1,000 each year and issued at par, cannot be determined

until we know at what rate the amounts were reinvested up to the date of the last maturity. Reinvestment has nothing to do with the yield of the original investment. Nevertheless, two authors have constructed tables based upon a dual rate, one a rate of income, the other a rate of accumulation, and they have taken the latter at the arbitrary figure of 4%, irrespective of the grade of the bond.

§ 329. Modified Method for Valuing Premiums

It is proper to give the method by which these results seem to be obtained, or, at least, a method which will produce those results.

As a preliminary we will consider the valuation of a premium in a slightly different way from any yet given.

We have seen that the premium on $1 is the present worth (at the income rate) of the difference of rates. We may modify this by saying that it is the difference of rates $(c - i) \times$ the present worth of an annuity of $1 (P), which may be found in Table IV.* But to multiply by P is the same thing as to divide by $1 \div \mathrm{P}$, or 1/P. Therefore, another expression for the premium is $(c - i) \div (1/\mathrm{P})$. But we found in § 90 that the rent (1/P) is the sum of the sinking fund (1/A) and the single interest (i). Therefore, we still further modify our expression:

$$\text{Premium} = (c - i) \div (i + 1/\mathrm{A})$$

§ 330. Rule for Valuation of a Premium

Rule: Subtract the income rate from the cash rate, and use this as a dividend. Add the instalment from Table V* to the rate of income, and this will be the divisor. The quotient will be the premium.

Example: What is the premium on a 6% bond (semi-annual) for $1, 50 years, yielding 5%?

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* In Chapter XXXII.

$c = .03$; $i = .025$; $c - i = .005$ (dividend)
1/A at 2½%, 100 periods = .002312 (Table V*)
.025 + .002312 = .027312 (divisor)
Premium = .005 ÷ .027312 = .18307
Value of bond, $1.18307

§ 331. Computation at Dual Rate

To introduce the feature of an accumulative rate differing from the income rate, it is only necessary to change one term in the above formula. The value of 1/A must be taken from the column of Table V,* which represents the accumulative rate, i remaining as the income rate.

Example: What is the premium on a 6% bond, as above, yielding 5% on the entire investment to maturity, the principal being replaced by a sinking fund at 4%?

$c - i = .03 - .025 = .005$ (dividend)
1/A at 2% = .003203 (Table V*)
$i + 1/A = .025 + .003203 = .028203$ (divisor)
.005 ÷ .028203 = .17729
Value of bond = $1.17729, agreeing with Croad's and Robinson's tables.

The constant income is .0294322 (i.e., 2½% of the value of the bond), which subtracted from the cash received .03, leaves as contribution to the sinking fund .0005678. At 4% an annuity of .0005678 will amount in 50 years to .17729||, as may be ascertained from Table III,* thus replacing the premium.

§ 332. Dual Rate in Bookkeeping

This form of valuation, which introduces an arbitrary element, cannot be satisfactorily applied in the bookkeeping processes of Chapter XVII. It is impossible to derive one value from another consistently. The result will not agree

* In Chapter XXXII.

with a fresh calculation, and the profit or loss on a sale will be distorted. Any intermediate value, as shown by the actuaries, may have three different versions.

§ 333. Utilization of Dual Principle

While tables on a fixed replacement rate are useless for purchasing securities, the principle may occasionally be utilized. Thus, the trustee referred to in § 148 may find that it is impracticable to invest favorably such small amounts as \$400 or \$500, and may conclude to deposit a sinking fund in a savings bank where he may reasonably expect that it will accumulate for the next five years at 3½%, or he may make a contract with a trust company on the same terms. He may then decide also that it is better for the beneficiary to receive a uniform income, rather than one gradually decreasing.

At 1¾% per period, a sinking fund of \$.092375 will, in ten periods, amount to \$1; therefore, by multiplication it will take a sinking fund of \$414.883 to accumulate to \$4,491.29 in 10 periods. Out of the coupon of \$2,500 must be taken the instalment of \$414.88, leaving for the beneficiary a constant semi-annual income of \$2,085.12, instead of the \$2,089.83 with which he would have begun on the replacement plan, and which would have gradually fallen to \$2,079.82 for the last half-year.

§ 334. Installation of Amortization Accounts

When the accounts of securities have once been established on the plan of gradual extinction of premiums and discounts, it is not difficult to take care of each new purchase as it comes in, and to prepare its appropriate schedule, running if desired all the way to the date of redemption. When, however, the accounts have been previously kept on the basis of par or of cost, and it is desired to introduce investment values instead, the task is much greater.

§ 335. Scope of Calculations

It might be supposed at first thought that it would be necessary to start the schedules back at the date of purchase, but this is entirely unnecessary. For example, we find a 5% bond for $100,000 which 20 years ago was bought for $112,650, and which has still 10 years to run. At the date of purchase it must have had 30 years to run. Turning to any table of 5% bonds, 30-year column, we find that this was (within 39 cents) a 4¼% basis. Turning to the 10-year column, it appears that the value of a 5-year bond at 4¼% is $106,058.46. It is sufficiently accurate to begin with this value, disregarding the 39 cents residue, although that residue might be eliminated by the proportion,

12,649.61 : 6,058.46 :: 39 : 19

This would increase the present value to $106,058.65.

§ 336. Method of Procedure when Same Basis Is Retained

So long as the same basis is preserved, any number of intervening years may be disregarded. The following procedure may be recommended:

(1) Make an accurate list of the issues held, giving the following particulars: dates of maturity; dates of purchase; par value of each lot; cost of each lot, being at the rate of $.... per $1,000 of par; rate of interest paid, and the income basis when ascertained. Leave a column for valuation at a date one period earlier than the proposed date of transformation.

(2) Ascertain on what income basis each lot was bought. This is done most easily by using the tables. In these and the subsequent calculations it will be found advantageous to use blank books and entrust nothing to loose papers. Head each calculation with a statement of the problem which it solves. Paper for these blank books, ruled with vertical lines, every third one of which is darker than the other two,

will much facilitate the work, and it is desirable to have the pages numbered in a continuous series, for reference.

(3) Find the value of each lot at the initial date, which is, as already stated, one period earlier than the date on which the books are to be transformed to investment values.

(4) Where different lots of the same class have been purchased at various dates and prices, their values at the various bases on the initial date should be added together, giving a composite value. Ascertain what is the income basis for the time yet to run on this composite value. This basis is the average basis for the remaining time of the bond.

(5) Having carefully verified all the initial values and the effective rates, proceed to calculate the amortization and accumulation of each class for one period, commencing a schedule for each. The resulting values should be again verified with care, these being the values with which the new accounts will begin.

(6) Continue the calculations of successive values, carrying them into decimals two places beyond the cents, ignoring slight differences in the last figure. Copy the results, rounded to the nearest cent, into the schedules, and complete the latter. If time allows, it is advisable to calculate each schedule to maturity, because no better proof of the correctness of the entire chain of values can be had than the fact that the bond reduces exactly to par at maturity. But if time presses, only a few of the values may be calculated, but the last one should be verified by some independent method. It is well in this case to leave in the blank book sufficient room to complete the calculations for each schedule. A reference on the schedule to the page of the blank book where the calculation is made, will be useful.

(7) Make such entries as will place the ledger or ledgers on the investment-value basis.

Part III—Logarithms

CHAPTER XXIX

FINDING A NUMBER WHEN ITS LOGARITHM IS GIVEN

§ 337. Logarithmic Tables

The meaning and use of logarithms have already been discussed in a general way,* and a simple three-figure, four-place table of logarithms given (§ 43). The expression "three-figure" refers to the number of figures in each of the numbers of the table, and the expression "four-place" refers to the number of decimals in each of the corresponding logarithms. In the table given, for example, the logarithm of 7.41 is shown to be .8698.

§ 338. Discussion of Logarithms

As previously explained, every logarithm consists of the characteristic, or whole number (which is frequently zero), and a decimal fraction. Occasionally the decimal fraction is zero, as in the case of the logarithms of .01, .1, 1, 10, 100, etc. The decimal fractions which constitute that part of the logarithm requiring tabulation are interminate; that is, their values may be computed to any desired number of decimal

* See Chapter III.

places and the last place will still be inexact. Thus, the logarithm of 2 to 20 places is .301 029 995 663 981 195 21+. In a 4-place table, this would be rounded

off to	.301 0
in a 7-place table, to	.301 030 0
in a 10-place table, to	.301 029 995 7
in a 12-place table, to	.301 029 995 664

The terminal decimal is never quite accurate, but is nearer the true value than either the next greater decimal or the next smaller one. Thus, the logarithm .8698 is nearer the true logarithm of 7.41 than either .8699 or .8697.

§ 339. Standard Tables of Logarithms

The tables most in use, like those of Vega, Chambers, and Babbage, are of five figures and seven places. A six-figure table would have to contain ten times as many logarithms as a five-figure table and, even though the number of *places* were not increased, the space occupied would be ten times greater than in the case of the five-figure table. In the tables above mentioned, two figures in addition to the five tabulated may be obtained by interpolation.

§ 340. United States Coast Survey Tables

The tables of the United States Coast Survey have five figures and ten places. Nine figures may be obtained by simple proportion, but the tenth is, for most purposes, unreliable.

It will, of course, be understood that the more decimal places given in the tables, the more figures we can obtain in the corresponding numbers, but the number of figures (in the desired number) can never be more than the number of places (in the corresponding logarithm). All of the foregoing tables give auxiliary tables of proportionate parts or differences.

§ 341. Gray and Steinhauser Tables

Tables of 24 and 20 places have been published by Peter Gray and Anton Steinhauser, respectively, but the plan for extending the number of figures is quite different from the method of simple interpolation above referred to. Both of these authors proceed on the plan of subdividing the number into factors, and adding together the logarithms of those factors.

§ 342. A Twelve-Place Table

For the accurate computation of problems in compound interest, specially designed tables will be found in Chapter XXX. A limit of twelve figures has been selected as the most useful for this purpose. In the logarithms tabulated, thirteen decimal places are given, the thirteenth place insuring the accuracy of the twelfth figure of the corresponding number, which would otherwise sometimes be 1, 2, or even 3 units in error, through the roundings being preponderant in one direction or the other.

§ 343. The "Factoring" Method

The method used in finding logarithms within the scope of these tables, but not directly given in them, is that of factoring, it being possible to construct the logarithm of any number of twelve figures or less (999,999,999,999 in all) by some combination of the 584 logarithms given in the table of factors (§ 358).

Column A contains the logarithms of numbers of two figures, 11 to 99, both inclusive, carried to thirteen places of decimals.

Column B contains the logarithms of four-figure numbers 1.001 to 1.099, each beginning with 1. and one zero.

Column C contains the logarithms of six-figure numbers 1.00001 to 1.00099, each beginning with 1. and three zeroes.

Column D, 1.0000001 to 1.0000099, beginning with 1. and five zeroes.

Column E, 1.000000001 to 1.000000099, beginning with 1. and seven zeroes.

Column F, 1.00000000001 to 1.00000000099, beginning with 1. and nine zeroes.

For example, opposite 34 in the table we find:

A	.531 478 917 042,3	*ln*	3.4
B	.014 520 538 757,9	*ln*	1.034
C	.000 147 635 027,3	*ln*	1.00034
D	.000 001 476 598,7	*ln*	1.0000034
E	.000 000 014 766,0	*ln*	1.000000034
F	.000 000 000 147,7	*ln*	1.00000000034

By omitting all the prefixed zeroes, the printed table is made very compact, each complete line across the table of factors shown in § 358 containing only 57 figures instead of 82, as would otherwise be necessary. In using the tables this must be taken into consideration, and accordingly it will be understood hereafter that C 34, for example, means the number 1.00034, and F 34 means 1.00000000034.

§ 344. Finding a Number from Its Logarithm

In this process there are two stages: first, to divide the logarithm into a number of partial logarithms taken from those contained in the table of factors; second, to multiply together the numbers corresponding to these logarithms. Of course only the decimal part of the logarithm is used, and the number has the position of its units figure determined from the characteristic of the logarithm.

Let the logarithm .753 797 472 366,5 be one which has been obtained as the result of an operation, and let the corresponding number be required. Search in Column A for the highest logarithm which does not exceed the given

one. This is found to be .748 188 027 006,2, which stands opposite 56.

Subtracting from	.753 797 472 366,5
A 56	.748 188 027 006,2
we have the remainder	5 609 445 360,3
This is smaller than any logarithm in Column A. We search for it in Column B and find opposite 13 precisely the same figures	5 609 445 360,3

These two logarithms added together make the given logarithm; hence the product of their numbers gives the number required.

To multiply 56 by 1.013:

$$\left.\begin{array}{l} 56 \\ \underline{1013} \\ 56 \\ \;\;56 \\ \;\;168 \\ \hline 56728 \end{array}\right\} \text{or} \left\{\begin{array}{ll} & \;\;1013 \\ \hline 5 & 5065 \\ 6 & \;\;6078 \\ \hline & 56728 \\ & \\ & \end{array}\right.$$

This process may be greatly simplified as follows, placing the figures of the multiplier in vertical order at the side:

$$\begin{array}{r|l} & 56 \\ 1 & \;\;56 \\ 3 & \;\;168 \\ \hline & 56728 \end{array} \qquad \text{or} \qquad \begin{array}{rl} & 56 \\ 13 \times 5 & \;065 \\ 13 \times 6 & \;\;\;078 \\ \hline & 56728 \end{array}$$

Notice that the first product is moved two columns to the right of the multiplicand.

The column G used in the following example is not given in the table of factors, but it is found by simply taking the first two figures from E. The "G" number in this case may be either 55 or 56, which may make the thirteenth figure of the result doubtful, but probably not the twelfth.

Now take a larger logarithm....		753 911 659 107,4
and continue the subtraction	A 56	748 188 027 006,2
		5 723 632 101,2
	B 13	5 609 445 360,3
		114 186 740,9
	C 26	112 901 888,7
		1 284 852,2
	D 29	1 259 452,2
		25 400,0
	E 58	25 189,1
		210,9
	F 48	208,5
		2,4
	G 55+	2,4

		5600	
	1	56	
	3	168	
		567280000	
(See Note 1*)	2	113456	
	6	340368	
		5674274928000	
(See Note 2*)		★	
	2	11348550	ǁ
	9	5106847	
		5674291383397	
(See Note 3*)	5	● 283715	
	8	● 45394	
	4	● 2270	
	8	● 454	
	5	● 28	
	5	3	
(See Note 4*)		567429171526	

* On following page.

Note 1: The second multiplication jumps its right-hand figure (6) *four* places to the right, which may be marked off by four zeroes, or four dots.

Note 2: Having extended the product to include the 13th figure, contraction begins in this multiplicand; its first figure used being the 7th (marked ★) allowing for the carrying from the 8th. Thus the starting point for this multiplication is moved *six* places *back*.

Note 3: The multiplicand need no longer be extended, as has been done at successive stages above, but remains the same to the end. For convenience, dots may be placed in advance under the first figure to be used in multiplication in each line.

Note 4: The thirteenth figures are added, but only used for carrying to the twelfth. In this example the total of the last column is 31, but it does not appear, except as contributing 3 to the next column.

The dot below a figure indicates where the contracted multiplication begins, all the figures to the right being ignored, except as to their carrying power.

§ 345. Procedure in an Unusual Case

Required the number for log. 011 253 170 127.

In this example there is no suitable logarithm in A and we must begin with B, as shown on page 290.

This example illustrates the procedure when B furnishes the first logarithm. It also shows the convenience of using paper ruled for the purpose.

In order to set down the partial products without hesitation, remember the numbers 2, 4, 6.

In multiplying by B, the first figure of the product moves two places to the right.

In multiplying by C, the first figure of the product moves four places to the right.

Formation of Number from Logarithm

Logarithm	0	1	1	2	5	3	1	7	0	1	2	7	0
A —													
B 26		1	1	1	4	7	3	6	0	7	7	5	8
				1	0	5	8	0	9	3	5	1	2
C 24				1	0	4	2	1	8	1	7	0	0
						1	5	9	1	1	8	1	2
D 36						1	5	6	3	4	5	7	3
								2	7	7	2	3	9
E 63								2	7	3	6	0	5
										3	6	3	4
F 83										3	6	0	5
												2	9
G 67												2	9
A —													
B	1	0	2	6									
26	1	0	2	6				★					
C 2					2	0	5	2					
4						4	1	0	4				
	1	0	2	6	2	4	6	2	4				
							●						
D 3							3	0	7	8	7	3	9
6						●		6	1	5	7	4	8
	1	0	2	6	2	4	9	9	3	4	4	8	7
E 6					●				6	1	5	7	5
3				●						3	0	7	9
F 8			●								8	2	1
3		●										3	1
G 6	●												6
7													1
	1	0	2	6	2	5	0	0	0	0	0	0	

In multiplying by D, the first figure of the *multiplicand* moves six places to the *left*.

The following rule may now be formulated for this process.

§ 346. Rule for Finding Number when Logarithm Is Given

(a) By successive subtractions separate the given logarithm into a series of partial logarithms found in the columns of the table of factors, setting opposite each its letter and number.

(b) By successive multiplications find the product of all the numbers thus found, allowing, in the placing of the partial products, for the prefixed 1 and zeroes.

The work may be made to occupy fewer lines by setting down the factors E, F, and G as one number at the top, multiplying it by A, and incorporating it thereafter as one multiplicand with the preceding figures. The result will not be affected. Let the factors be, as in the first example: A 56, B 13, C 26, D 29, E 58, F 48, and G 55.

```
                     E F G
                    584855
                   -------
A 56               2924275
                    350913
      -------------
      5600000327519
B 13    56000003275
        16800000982
      -------------
      5672800331776
C 26     1134560066
          340368029
      -------------
      5674275259871
D 29        11348551
             5106848
      -------------
      5674291715270
```

Required the number whose logarithm is .5 or ½.

	.500 000 000 000,0
A 31	491 361 693 834,3
	8 638 306 165,7
B 20	8 600 171 761,9
	38 134 403,8
C 08	34 742 168,9
	3 392 234,9
D 78	3 387 483,7
	4 751,2
E 10	4 342,9
	408,3
F 94	408,2
G 03	0,1

The resulting factors, A 31, B 20, C 08, D 78, E 10, F 94, and G 03, when combined produce the result 3.1 6 2 2 7 7 6 6 0 1 7. The multiplication illustrates how zeroes are treated when they occur in the multipliers. The result is the square root of 10, to 12 places, as may be demonstrated by multiplying 3.1 6 2 2 7 7 6 6 0 1 7 by itself.

§ 347. Method by Multiples

In order to facilitate the multiplication of the factors, A, B, C, etc., the table of multiples* (§ 361), giving the product of each number from 1 to 9, by every number from 2 to 99, will be found convenient. Thus, the multiples of 89 read in one line as follows:

* Devised by Arthur S. Little.

1	2	3	4	5	6	7	8	9
089	178	267	356	445	534	623	712	801

Then, if it be desired, for example, to multiply 68792341 by 89, we would select from the above table

```
under 6   534
      8    712
      7     623
      9      801
      2       178
      3        267
      4         356
      1          089
          __________
          6122518349
```

We have thus multiplied each figure of the multiplicand by both figures of the multiplier, setting down each partial product unhesitatingly. Three figures must be set down for each partial product, even if the first be a zero. The work may be made more compact by piling the partial products like bricks, using only three lines:

```
534,801,356,
 712,178,089
  623,267,
__________
6122518349
```

To use this method in combining the factors of a number, the letters A, B, C, etc., are written above alternate figure spaces, which is facilitated by the use of paper properly ruled. Then the first partial product under each letter is placed with its middle figure under that letter at the top.

The following is an example of a combination already performed in another form:

	A B C D E F G
A 56	1 584855
	280,448
	448,280
	224,28
	56 327519
B 13	065 039,06
	078 026,2
	091
	5672800331778
C 26	130,052,000,
	156,208,078
	182,000,08
	5674275259864
D 29	145,116,15
	174,058,1
	203,203
	567429171526

A process* for verifying a numerical result, by using a different set of factors in a second operation, is as follows:

Required the number corresponding to

.305 773 384 163,0

The factors are A 20, B 10, C 97, D 21, E 94, F 94, and G 33; and the number is 2.02196383809.

In order to check the result and make sure of perfect accuracy, we may solve the problem a second time, using a smaller factor for A, provided the first remainder be less than B 99, or .040997692423,5. Using A 19 instead of A 20:

* Suggested by Arthur S. Little.

	305 773 384 163,0
A 19	278 753 600 952,8
	27 019 783 210,2
B 64	26 941 627 959,0
	78 155 251,2
C 17	73 823 787,1
	4 331 464,1
D 99	4 299 494,1
	31 970,0
E 73	31 703,5
	266,5
F 61	264,9
G 37	1,6

The new factors are A 19, B 64, C 17, D 99, E 73, F 61, and G 37.

By multiplication, we obtain the same result as before:

A 19		7 361,37
		6 625,233
	19	13 986,60
B 64	11 4	839,2
	76	55,9
		202 160 014 881,7
C 17		20 216 001,5
		14 151 201,1
		202 194 382 084,3
D 99		1 819 749,5
		181 974,9
		202 196 383 809

CHAPTER XXX

FORMING LOGARITHMS; TABLES

§ 348. Explanation of Process

To form the logarithm of a given number—the table of factors being used—two processes are necessary: first, the number is separated into a series of factors corresponding to the six columns of the thirteen-place table; second, the logarithms of these factors are taken from the table and added together.

The factoring is effected by a progressive division, as illustrated by the following simple example:

To find the logarithmic factors, A, B, C, etc., of 5.6728. First extend the number to 12 places, 567 280 000 000. The first factor, A, is always the first two figures of the number itself.

```
A 56)56 7 2 80 000 000 (1.013 B
     56
     --
        7 2
        5 6
        -----
        1 6 8
        1 6 8
        -----
```

It will readily be seen that one 56 might have been omitted.

```
A 56)7 280 000 000 (B 13
     5 6
     ----
     1 68
     1 68
     ----
```

Turning then to the table, we have only to set down the logarithms of these two factors:

A 56	*nl*	748 188 027 006,2
B 13	*nl*	5 609 445 360,3
56728	*nl*	753 797 472 366 5

B 13 may be regarded as an abbreviation of 1.013.

In the next example a second divisor, at least, is required.

```
      A 56) 7 4 2 9 1 7 1 5 2 6 (B 13
            5 6
            -----
            1 8 2
            1 6 8
            -----
AB 56 728 ) 1 4
```

The second divisor is the product of A and B. It might be obtained in either of three ways:

By multiplication $56 \times 1.013 = 56728$

By addition

$$\begin{array}{rr} & 56 \\ + & 56 \\ + & 168 \\ \hline & 56728 \end{array}$$

But the easiest way is

by subtraction 56742 (first five figures of the number)

$$\begin{array}{rr} & 56742 \\ - & 14 \\ \hline & 56728 \end{array}$$

(14 = the remainder)

This is the proper method for forming all divisors after the first; that is, subtract the remainder from the original number so far as used.

We resume the division, bringing down four more figures, to the ninth inclusive:

```
AB ) 56728      )149171526(C26
                 113456
                 ------
                  357155
                  340368
                  ------
ABC ) 5674275) ★1678726(D29
                 1134855
                 -------
                  543871
                  510685
                  ------
      56742914     33186(E58
                   28371
                   -----
                    4815
                    4539
                    ----
                     276(F48,7
                     227
                     ---
                      49
                      45
                      --
                       4
```

The third divisor A B C is also formed by subtracting

```
from the number     567429171 5
★ the remainder          16787
                    -----------
   leaving          5674274928
```

As only six figures are needed for the divisor and one additional figure for carrying, this is rounded up to

5 6 7 4 2 7,5

The fourth divisor is practically the number itself so far as needed, and this lasts to the end.

The entire process is now repeated, but to insure greater

accuracy in the twelfth figure we will divide out to the thirteenth:

```
                                  A 56) 7 4 2 9 1 7 1 5 2 6,0 (B 13
                                        5 6
                                        ---
                                        1 8 2
                                        1 6 8
                                        -----
                         A B 56 728)      1 4 9 1 7 1         (C 26
                                          1 1 3 4 5 6
                                          -----------
                                            3 5 7 1 5 5
                                            3 4 0 3 6 8
                                            -----------
            A B C 56 742 749 28)              1 6 7 8 7 2 6,0 (D 29
(Contracted division begins here)             1 1 3 4 8 5 5,0
                                              ---------------
                                                5 4 3 8 7 1,0
                                                5 1 0 6 8 4,7
                                                -------------
                         56 742 92)               3 3 1 8 6,3 (E 58
                                                  2 8 3 7 1,4
                                                  -----------
                                                    4 8 1 4,9
                                                    4 5 3 9,4
                                                    ---------
                                                      2 7 5,5 (F 48
                                                      2 2 7,0
                                                      -------
                                                        4 8 5
                                                        4 5 4
                                                        -----
                                                          3 1 (G 55
                                                          2 8
                                                          ---
                                                            3
```

It remains only to add together the logarithms:

A	56	(*nl*)	748 188 027 006,2
B	13	"	5 609 445 360,3
C	26	"	112 901 888,7
D	29	"	1 259 452,2
E	58	"	25 189,1
F	48	"	208,5
G	55	"	2,4
567 429 171 526		(*nl*)	753 911 659 107

The figures in the thirteenth column are used only for carrying to the twelfth column.

§ 349. Rule for Finding a Logarithm

We may now formulate the following rule for finding the logarithm:

(a) Fix the number at 13 figures, by adding ciphers or cutting off decimals.

(b) Cut off the two left-hand figures by a curve, giving A.

(c) Divide the next three figures by A, giving the two figures of B, and a remainder.

(d) Form the second divisor A B, by subtracting the remainder from the first five figures of the number.

(e) Bring down four more figures to the remainder and divide by A B, giving the two figures of C and a remainder.

(f) Form the third (and last) divisor A B C by subtracting the remainder from ten figures of the number.

(g) Divide the remaining figures by the third divisor. As there are ten figures in the divisor and only eight in the dividend, contraction begins immediately. Having obtained the figures of D, the divisor for E, F, and G is simply the number itself contracted.

(h) Write down the logarithms of A, B, C, D, E, and F, obtained from the several columns of the table of factors; also that of G, being the first two figures of the corresponding E. The sum will be the mantissa or decimal part of the logarithm of the number, the thirteenth decimal place being used for carrying only.

§ 350. Examples of Logarithmic Computations

It is advisable, for the sake of both convenience and accuracy, to make all of these logarithmic computations on paper ruled with at least thirteen vertical lines, every third line being darker than the other two. Space should be left on either side of these lines for writing in the divisors and quotients, and for such other arithmetical work as may be necessary. As a rule, however, there would be few, if any, additional arithmetical computations which would have to be performed at the sides.

A few examples for practice are given below with the factors and the solution:

5674=A 56 B 13 C 21 D 15 E 35 F 42 G 70
log. 5674=3.753 889 331 458
38.8586468578 =A 38 B 22 C 58 D 31 E 39 F 02 G 25
log. do. =1.589 487 673 453
3.1415926535898+=A 31 B 13 C 41 D 16 E 33 F 11 G 91
log. do. =.497 149 872 694

(This number is the ratio of the circumference of a circle to its diameter.)

1.02625=B 26 C 24 D 36 E 63 F 83
log. do. = .011 253 170 127

This number begins with an expression of the form B (1.026), hence no division by A occurs. 1026 is the first divisor.

```
B   1026) 2 5 0 0                   C 24
          2 0 5 2
          -------
            4 4 8 0
            4 1 0 4
            -------
B C 102624624)  3 7 6 0 0 0,0   D 36
                3 0 7 8 7 3,9
                -------------
                  6 8 1 2 6,1
                  6 1 5 7 4,8
                  -----------
                    6 5 5 1,3   E 63
       102625)      6 1 5 7,5
                    ---------
                      3 9 3,8
                      3 0 7,9
                      -------
                        8 5,9   F 83
                        8 2,1
                        -----
                          3,8
                          3,1
                          ---
                            7   G 70
B 26    011 147 360 775,8
C 24        104 218 170,0
D 36          1 563 457,3
E 63             27 360,6
F 83                360,5
G 70                  3,0
        -----------------
```

log. 1.02626 = .011 253 170 127*

§ 351. Logarithms to Less Than Twelve Places

The table of factors may be cut down to any lower number of places. In the example in § 348 it may be required

* This result will be found also in the Table of Interest Ratios, but even more extended.

to give 9 places only, the tenth being used for carrying. We cut down the original logarithm to ten figures, with a comma after the ninth, and it becomes:

	753 911 659,1
A 56	748 188 027,0
	5 723 632,1
B 13	5 609 445,4
	114 186,7
C 26	112 901,9
	1 284,8
D 29	1 259,5
	25,3
E 58	25,2
F 24	1

A	5 6
B 1	5 6
3	1 6 8
	5 6 7 2 8 0 0 0 0
C 2	1 1 3 4 5 6
6	3 4 0 3 6,8
	5 6 7 4 2 7 4 9 2,8
D 2	1 1 3 4,9
9	5 1 0,7
E 5	2 8,4
8	4,5
F 2	1
	5 6 7 4 2 9 1 7 1,4

The number is slightly in error in its tenth place, but correct to the ninth.

§ 352. Tables with More Than Twelve Places

If a table of factors for 18 or some other number of places should hereafter be prepared, the methods which have been explained would be applicable to the new table.

§ 353. Multiplying Up

Another method for obtaining the factors of the number in forming its logarithm* proceeds by multiplication instead of division, the latter operation being notably the more laborious. The number, at first taken as a decimal less than 1, is successively multiplied up to produce 1.000,000,000,000,0 and these multipliers are the A, B, C, D, E, F, and G, whose logarithms added together make the cologarithm, or logarithm of the reciprocal, from which the logarithm is easily obtained.

§ 354. Process of Multiplying Up

A is a number of two figures, a little less than the reciprocal of the number, which will be called the sub-reciprocal of its two initial figures. A table of sub-reciprocals is given in § 360. The number multiplied by A will always give a product beginning with 9. B is always the arithmetical complement of the two figures following the nine, or the remainder obtained by subtracting those two figures from 99. Multiplication by B will usually give a result beginning with 999. C is the next complement and gives five 9's, 999,99. D similarly brings 999,999,9**,***,*. No further multiplication is necessary, after D has been used as a factor; the six figures in the places of the asterisks are the complements of E, F, and G.

To illustrate, let it be required to obtain the logarithm to the 12th place of .314 159 265 359 0. The object is to multiply .314 159 265 359 up to 1.000 000 000 000 0. The

* Suggested by Edward S. Thomas of Cincinnati.

first step is to find the sub-reciprocal of 31, or A. Turning to the table of sub-reciprocals, opposite 31 we find 31, by which we multiply.

		.3141592653590	
	A 31	.9424777960770	
		314159265359	
		.9738937226129	(One nine secured)
99 — 73 = 26 B 26 is therefore the next multiplier; dropping the last two figures		194778744523	
		58433623357	
		.9992149594009	(Three nines secured)
(99—21)	C 78	6994504716	
		799371968	
		.9999943470693	(Five nines secured)
(99—43)	D 56	49999718	
		5999966	
		.9999999470377	(Seven nines secured)
(99—47)	E 52	52	
(99—03)	F 96	96	
(100—77)	G 23	23	

A 31 *nl*	.4913616938343
B 26	111473607758
C 78	3386176522
D 56	24320423
E 52	225833
F 96	4169
G 23	10
colog.	0.502850127306
log.	$\bar{1}$.497149872694

§ 355. **Supplementary Multiplication**

It may happen, in the course of multiplication, that the complement of the figures following the 9 does not suffice to secure two nines more. In this case, another supplementary multiplication must take place. This occurs in the following example, which has already been solved in § 348.

Required the logarithm of the number

.567 429 171 526

In this example the C multiplication also requires an additional figure. This seldom occurs.

	.567 429 171 526 0
A 17	.397 200 420 068 2
	.964 629 591 594 2
B 35	28 938 887 747 8
	4 823 147 958 0
	.998 391 627 300 0
B 01	998 391 627 3
	.999 390 018 927 3
C 60	599 634 011 4
	.999 989 652 938 7
C 01	9 999 896 5
	.999 999 652 835 2
D 03	299 999 9
	.999 999 952 835 1
E 47, F 16, G 49	47 164 9

A 17	230 448 921 378 3
B 35	14 940 349 792 9
01	434 077 479 3
C 60	260 498 547 4
01	4 342 923 1
D 03	130 288 3
E 47	20 411 8
F 16	69 5
G 49	2 1
colog.	.246 088 340 892 7
log.	$\bar{1}$.753 911 659 107 3

As the multiplication by B 35 brings only 998 instead of 999, we multiply again by B 01, which brings it up to 999+.

In the next example there is a large defect in the product obtained by multiplying by B 85, which requires an additional multiplication by B 7.

	110 175*	
A 83	881 400	(83, sub-reciprocal of 11)
	33 052 5	
	914 452 5	
B 85	73 156 200	
	4 572 262 5	
	992 180 962 5	
B 07	6 945 266 737,5	
	999 126 229 237,5	
C 87	799 300 983,4	
	69 938 836,0	
	999 995 469 056,9	
D 45	3 999 981,9	
	499 997,7	
	999 999 969 036,5	
E 30, F 96, G 35	30 963,5	
A 83	919 078 092 376,1	
B 85	35 429 738 184,5	
B 07	3 029 470 553,6	
C 87	377 671 935,8	
D 45	1 954 320,8	
E 30	13 028,8	
F 96	416,9	
G 35	1,5	
colog.	.957 916 940 818 0	
log.	$\bar{1}$.042 083 059 182 0	

*The number 110 175 was purposely selected, very slightly in excess of the highest number in column B, so as to produce the shortage of 7.

§ 356. Multiplying Up by Little's Table

	.137 128 857 423 9	
A 71	0 710 715 682 846 4	(71 being the
	213 142 355 142 0	sub-reciprocal
	49 756 849 721 3	of 13.)
	.973 614 887 709 7	
B 26	23 415 620 818 2	
	1 820 262 080 0	
	078 104 182 2	
	.998 928 874 790 1	
B 01	998 928 874 8	
	.999 927 803 664 9	
C 07	69 994 946 3	
	.999 997 798 611 2	
D 22	1 981 981 8	
	198 198 0	
	19 815 4	
	.999 999 998 606 4	
	01 393 6	
	E F G	
A 71	851 258 348 719 1	
B 26	11 147 360 775 8	
B 01	434 077 479 3	
C 07	30 399 549 8	
D 22	955 446 8	
E 01	434 3	
F 39	169 4	
G 36	1 6	
	862 871 142 576 1	
	.137 128 857 423 9	

which is the log. of 1.371 288 574 239

In the preceding example, Little's table of multiples (§ 361) is used in the multiplication. It will be found that the logarithm when computed has the same figures as the number itself—a remarkable peculiarity which no other combination of figures can possess.

§ 357. Different Bases

Ten is the base of the logarithmic system which we have been explaining; it is the most useful of all systems, because ten is also the base of our numerical system. These are usually called common, or vulgar, or Briggsian logarithms, but decimal logarithms would seem a more appropriate name.

Any number might form the base of a system of logarithms, but the only other in actual use is one known as the "natural" system, having for its base the number 2.718281828459+, known to mathematicians as *e*, which is the sum of the series,

$$1 + \frac{1}{1} + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \frac{1}{1 \times 2 \times 3 \times 4} + \frac{1}{1 \times 2 \times 3 \times 4 \times 5} \text{ etc.}$$

This is only used in theoretical inquiries, and is seldom of utility to the accountant.

§ 358. TABLE OF FACTORS

No.	A * · *	B 1.0_{**}	C 1.000_{**}	D 1.00000_{**}	E $1.0^{7}{}_{**}$	F $1.0^{9}{}_{**}$	No.
		.0	.000	$.0^{5}$	$.0^{7}$	$.0^{9}$	
01	....	00 434 077 479,3	004 342 923,1	0 043 429,4	00 434,3	004,3	01
02	....	00 867 721 531,2	008 685 802,8	0 086 858,9	00 868,6	008,7	02
03	....	01 300 933 020,4	013 028 639,0	0 130 288,3	01 302,9	013,0	03
04	....	01 733 712 809,0	017 371 431,8	0 173 717,8	01 737,2	017,4	04
05	....	02 166 061 756,5	021 714 181,2	0 217 147,2	02 171,5	021,7	05
06	....	02 597 980 719,9	026 056 887,2	0 260 576,6	02 605,8	026,1	06
07	....	03 029 470 553,6	030 399 549,8	0 304 006,0	03 040,1	030,4	07
08	....	03 460 532 109,5	034 742 168,9	0 347 435,4	03 474,4	034,7	08
09	....	03 891 166 236,9	039 084 744,6	0 390 864,9	03 908,7	039,1	09
10	....	04 321 373 782,6	043 427 276,9	0 434 294,3	04 342,9	043,4	10
11	041 392 685 158,2	04 751 155 591,0	047 769 765,7	0 477 723,7	04 777,2	047,8	11
12	079 181 246 047,6	05 180 512 503,8	052 112 211,2	0 521 153,1	05 211,5	052,1	12
13	113 943 352 306,8	05 609 445 360,3	056 454 613,2	0 564 582,5	05 645,8	056,5	13
14	146 128 035 678,2	06 037 954 997,3	060 796 971,8	0 608 011,8	06 080,1	060,8	14
15	176 091 259 055,7	06 466 042 249,2	065 139 287,0	0 651 441,2	06 514,4	065,1	15
16	204 119 982 655,9	06 893 707 947,9	069 481 558,7	0 694 870,6	06 948,7	069,5	16
17	230 448 921 378,3	07 320 952 922,7	073 823 787,1	0 738 300,0	07 383,0	073,8	17
18	255 272 505 103,3	07 747 778 000,7	078 165 972,0	0 781 729,4	07 817,3	078,2	18
19	278 753 600 952,8	08 174 184 006,4	082 508 113,5	0 825 158,7	08 251,6	082,5	19
20	301 029 995 664,0	08 600 171 761,9	086 850 211,6	0 868 588,1	08 685,9	086,9	20
21	322 219 294 733,9	09 025 742 086,9	091 192 266,3	0 912 017,5	09 120,2	091,2	21
22	342 422 680 822,2	09 450 895 798,7	095 534 277,6	0 955 446,8	09 554,5	095,5	22
23	361 727 836 017,6	09 875 633 712,2	099 876 245,5	0 998 876,2	09 988,8	099,9	23
24	380 211 241 711,6	10 299 956 639,8	104 218 170,0	1 042 305,5	10 423,1	104,2	24

25	397 940 008 672,0	10 723 865 391,8	108 560 051,0	1 085 734,8	10 857,4	108,6	25
26	414 973 347 970,8	11 147 360 775,8	112 901 888,7	1 129 164,2	11 291,7	112,9	26
27	431 363 764 159,0	11 570 443 597,3	117 243 682,9	1 172 593,5	11 726,0	117,3	27
28	447 158 031 342,2	11 993 114 659,3	121 585 433,8	1 216 022,8	12 160,2	121,6	28
29	462 397 997 899,0	12 415 374 762,4	125 927 141,2	1 259 452,2	12 594,5	125,9	29
30	477 121 254 719,7	12 837 224 705,2	130 268 805,2	1 302 881,5	13 028,8	130,3	30
31	491 361 693 834,3	13 258 665 283,5	134 610 425,9	1 346 310,8	13 463,1	134,6	31
32	505 149 978 319,9	13 679 697 291,2	138 952 003,1	1 389 740,1	13 897,4	139,0	32
33	518 513 939 877,9	14 100 321 519,6	143 293 536,9	1 433 169,4	14 331,7	143,3	33
34	531 478 917 042,3	14 520 538 757,9	147 635 027,3	1 476 598,7	14 766,0	147,7	34
35	544 068 044 350,3	14 940 349 792,9	151 976 474,3	1 520 028,0	15 200,3	152,0	35
36	556 302 500 767,3	15 359 755 409,2	156 317 878,0	1 563 457,3	15 634,6	156,3	36
37	568 201 724 067,0	15 778 756 389,0	160 659 238,2	1 606 886,6	16 068,9	160,7	37
38	579 783 596 616,8	16 197 353 512,4	165 000 555,0	1 650 315,9	16 503,2	165,0	38
39	591 064 607 026,5	16 615 547 557,2	169 341 828,4	1 693 745,2	16 937,5	169,4	39
40	602 059 991 328,0	17 033 339 298,8	173 683 058,5	1 737 174,5	17 371,8	173,7	40
41	612 783 856 719,7	17 450 729 510,5	178 024 245,1	1 780 603,7	17 806,1	178,1	41
42	623 249 290 397,9	17 867 718 963,5	182 365 388,3	1 824 033,0	18 240,4	182,4	42
43	633 468 455 579,6	18 284 308 426,5	186 706 488,2	1 867 462,3	18 674,7	186,7	43
44	643 452 676 486,2	18 700 498 666,2	191 047 544,7	1 910 891,5	19 109,0	191,1	44
45	653 212 513 775,3	19 116 290 447,1	195 388 557,7	1 954 320,8	19 543,3	195,4	45
46	662 757 831 681,6	19 531 684 531,3	199 729 527,4	1 997 750,0	19 977,5	199,8	46
47	672 097 857 935,7	19 946 681 678,8	204 070 453,7	2 041 179,3	20 411,8	204,1	47
48	681 241 237 375,6	20 361 282 647,7	208 411 336,6	2 084 608,5	20 846,1	208,5	48
49	690 196 080 028,5	20 775 488 193,6	212 752 176,1	2 128 037,7	21 280,4	212,8	49

TABLE OF FACTORS—(*Continued*)

No.	A * · *	B 1.0**	C 1.000**	D 1.00000**	E 1.0^7**	F 1.0^9**	No.
		.0	.000	$.0^5$	$.0^7$	$.0^9$	
50	698 970 004 336,0	21 189 299 069,9	217 092 972,2	2 171 467,0	21 714,7	217,1	50
51	707 570 176 097,9	21 602 716 028,2	221 433 725,0	2 214 896,2	22 149,0	221,5	51
52	716 003 343 634,8	22 015 739 817,7	225 774 434,3	2 258 325,4	22 583,3	225,8	52
53	724 275 869 600,8	22 428 371 185,5	230 115 100,3	2 301 754,7	23 017,6	230,2	53
54	732 393 759 823,0	22 840 610 876,5	234 455 722,9	2 345 183,9	23 451,9	234,5	54
55	740 362 689 494,2	23 252 459 633,7	238 796 302,1	2 388 613,1	23 886,2	238,9	55
56	748 188 027 006,2	23 663 918 197,8	243 136 837,9	2 432 042,3	24 320,5	243,2	56
57	755 874 855 672,5	24 074 987 307,4	247 477 330,3	2 475 471,5	24 754,8	247,5	57
58	763 427 993 562,9	24 485 667 699,2	251 817 779,4	2 518 900,7	25 189,1	251,9	58
59	770 852 011 642,1	24 895 960 107,5	256 158 185,1	2 562 329,9	25 623,4	256,2	59
60	778 151 250 383,6	25 305 865 264,8	260 498 547,4	2 605 759,1	26 057,7	260,6	60
61	785 329 835 010,8	25 715 383 901,3	264 838 866,3	2 649 188,3	26 492,0	264,9	61
62	792 391 689 498,3	26 124 516 745,5	269 179 141,9	2 692 617,4	26 926,3	269,3	62
63	799 340 549 453,6	26 533 264 523,3	273 519 374,0	2 736 046,6	27 360,6	273,6	63
64	806 179 973 983,9	26 941 627 959,0	277 859 562,8	2 779 475,8	27 794,8	277,9	64
65	812 913 356 642,9	27 349 607 774,8	282 199 708,3	2 822 905,0	28 229,1	282,3	65
66	819 543 935 541,9	27 757 204 690,6	286 539 810,3	2 866 334,1	28 663,4	286,6	66
67	826 074 802 700,8	28 164 419 424,5	290 879 869,0	2 909 763,3	29 097,7	291,0	67
68	832 508 912 706,2	28 571 252 692,5	295 219 884,3	2 953 192,4	29 532,0	295,3	68
69	838 849 090 737,3	28 977 705 208,8	299 559 856,2	2 996 621,6	29 966,3	299,7	69
70	845 098 040 014,3	29 383 777 685,2	303 899 784,8	3 040 050,7	30 400,6	304,0	70
71	851 258 348 719,1	29 789 470 831,9	308 239 670,0	3 083 479,9	30 834,9	308,3	71
72	857 332 496 431,3	30 194 785 356,8	312 579 511,8	3 126 909,0	31 269,2	312,7	72
73	863 322 860 120,5	30 599 721 966,0	316 919 310,3	3 170 338,1	31 703,5	317,0	73
74	869 231 719 731,0	31 004 281 363,5	321 259 065,4	3 213 767,3	32 137,8	321,4	74

75	875 061 263 391,7	31 408 464 251,6	325 598 777,1	3 257 196,4	32 572,1	325,7	75
76	880 813 592 280,8	31 812 271 330,4	329 938 445,5	3 300 625,5	33 006,4	330,1	76
77	886 490 725 172,5	32 215 703 298,0	334 278 070,5	3 344 054,6	33 440,7	334,4	77
78	892 094 602 690,5	32 618 760 850,7	338 617 652,2	3 387 483,7	33 875,0	338,7	78
79	897 627 091 290,4	33 021 444 682,9	342 957 190,4	3 430 912,9	34 309,3	343,1	79
80	903 089 986 991,9	33 423 755 486,9	347 296 685,4	3 474 342,0	34 743,6	347,4	80
81	908 485 018 878,6	33 825 693 953,3	351 636 136,9	3 517 771,1	35 177,9	351,8	81
82	913 813 852 383,7	34 227 260 770,6	355 975 545,1	3 561 200,2	35 612,1	356,1	82
83	919 078 092 376,1	34 628 456 625,3	360 314 910,0	3 604 629,2	36 046,4	360,5	83
84	924 279 286 061,9	35 029 282 202,4	364 654 231,5	3 648 058,3	36 480,7	364,8	84
85	929 418 925 714,3	35 429 738 184,5	368 993 509,6	3 691 487,4	36 915,0	369,2	85
86	934 498 451 243,6	35 829 825 252,8	373 332 744,4	3 734 916,5	37 349,3	373,5	86
87	939 519 252 618,6	36 229 544 086,3	377 671 935,8	3 778 345,6	37 783,6	377,8	87
88	944 482 672 150,2	36 628 895 362,2	382 011 083,8	3 821 774,6	38 217,9	382,2	88
89	949 390 006 644,9	37 027 879 755,8	386 350 188,6	3 865 203,7	38 652,2	386,5	89
90	954 242 509 439,3	37 426 497 940,6	390 689 249,9	3 908 632,7	39 086,5	390,9	90
91	959 041 392 321,1	37 824 750 588,3	395 028 267,9	3 952 061,8	39 520,8	395,2	91
92	963 787 827 345,6	38 222 638 368,7	399 367 242,6	3 995 490,9	39 955,1	399,6	92
93	968 482 948 553,9	38 620 161 949,7	403 706 173,9	4 038 919,9	40 389,4	403,9	93
94	973 127 853 599,7	39 017 321 997,4	408 045 061,8	4 082 348,9	40 823,7	408,2	94
95	977 723 605 288,8	39 414 119 176,1	412 383 906,5	4 125 778,0	41 258,0	412,6	95
96	982 271 233 039,6	39 810 554 148,4	416 722 707,7	4 169 207,0	41 692,3	416,9	96
97	986 771 734 266,2	40 206 627 574,7	421 061 465,6	4 212 636,0	42 126,6	421,3	97
98	991 226 075 692,5	40 602 340 114,1	425 400 180,2	4 256 065,1	42 560,9	425,6	98
99	995 635 194 597,5	40 997 692 423,5	429 738 851,4	4 299 494,1	42 995,2	430,0	99

§ 359.

Table of Interest Ratios

$1 + i$	Logarithm	$1 + i$	Logarithm
1.00125	000 542 529 092 294	1.01	004 321 373 782 643
1.0015	000 650 953 629 595	1.01025	004 428 859 114 686
1.00175	000 759 351 104 737	1.0105	004 536 317 851 323
1.002	000 867 721 531 227	1.01075	004 643 750 005 712
1.00225	000 976 064 922 559	1.011	004 751 155 591 001
1.0025	001 084 381 292 220	1.01125	004 858 534 620 329
1.00275	001 192 670 653 684	1.0115	004 965 887 106 823
1.003	001 300 933 020 418	1.01175	005 073 213 063 604
1.00325	001 409 168 405 876	1.012	005 180 512 503 780
1.0035	001 517 376 823 504	1.01225	005 287 785 440 451
1.00375	001 625 558 286 737	1.0125	005 395 031 886 706
1.004	001 733 712 809 001	1.01275	005 502 251 855 626
1.00425	001 841 840 403 709	1.013	005 609 445 360 280
1.0045	001 949 941 084 268	1.01325	005 716 612 413 731
1.00475	002 058 014 864 072	1.0135	005 823 753 029 028
1.005	002 166 061 756 508	1.01375	005 930 867 219 212
1.00525	002 274 081 774 949	1.014	006 037 954 997 317
1.0055	002 382 074 932 761	1.01425	006 145 016 376 364
1.00575	002 490 041 243 299	1.0145	006 252 051 369 365
1.006	002 597 980 719 909	1.01475	006 359 059 989 323
1.00625	002 705 893 375 925	1.015	006 466 042 249 232
1.0065	002 813 779 224 673	1.01525	006 572 998 162 075
1.00675	002 921 638 279 469	1.0155	006 679 927 740 826
1.007	003 029 470 553 618	1.01575	006 786 830 998 449
1.00725	003 137 276 060 415	1.016	006 893 707 947 900
1.0075	003 245 054 813 147	1.01625	007 000 558 602 125
1.00775	003 352 806 825 089	1.0165	007 107 382 974 057
1.008	003 460 532 109 506	1.01675	007 214 181 076 625
1.00825	003 568 230 679 656	1.017	007 320 952 922 745
1.0085	003 675 902 548 784	1.01725	007 427 698 525 323
1.00875	003 783 547 730 127	1.0175	007 534 417 897 258
1.009	003 891 166 236 911	1.01775	007 641 111 051 437
1.00925	003 998 758 082 352	1.018	007 747 778 000 740
1.0095	004 106 323 279 658	1.01825	007 854 418 758 035
1.00975	004 213 861 842 026	1.0185	007 961 033 336 183

TABLE OF INTEREST RATIOS—(*Continued*)

1 + i	Logarithm	1 + i	Logarithm
1.01875	008 067 621 748 033	1.0275	011 781 830 548 107
1.019	008 174 184 006 426	1.02775	011 887 485 452 387
1.01925	008 280 720 124 194	1.028	011 993 114 659 257
1.0195	008 387 230 114 159	1.02825	012 098 718 181 213
1.01975	008 493 713 989 132	1.0285	012 204 296 030 743
1.02	008 600 171 761 918	1.02875	012 309 848 220 326
1.02025	008 706 603 445 309	1.029	012 415 374 762 433
1.0205	008 813 009 052 089	1.02925	012 520 875 669 524
1.02075	008 919 388 595 035	1.0295	012 626 350 954 050
1.021	009 025 742 086 910	1.02975	012 731 800 628 455
1.02125	009 132 069 540 472	1.03	012 837 224 705 172
1.0215	009 238 370 968 466	1.0305	013 047 996 115 232
1.02175	009 344 646 383 631	1.031	013 258 665 283 517
1.022	009 450 895 798 694	1.0315	013 469 232 309 170
1.02225	009 557 119 226 374	1.032	013 679 697 291 193
1.0225	009 663 316 679 379	1.0325	013 890 060 328 439
1.02275	009 769 488 170 411	1.033	014 100 321 519 621
1.023	009 875 633 712 160	1.0335	014 310 480 963 307
1.02325	009 981 753 317 307	1.034	014 520 538 757 924
1.0235	010 087 846 998 524	1.0345	014 730 495 001 753
1.02375	010 193 914 768 475	1.035	014 940 349 792 937
1.024	010 299 956 639 812	1.0355	015 150 103 229 471
1.02425	010 405 972 625 180	1.036	015 359 755 409 214
1.0245	010 511 962 737 214	1.0375	015 988 105 384 130
1.02475	010 617 926 988 539	1.038	016 197 353 512 439
1.025	010 723 865 391 773	1.039	016 615 547 557 177
1.02525	010 829 777 959 522	1.04	017 033 339 298 780
1.0255	010 935 664 704 385	1.041	017 450 729 510 536
1.02575	011 041 525 638 950	1.0425	018 076 063 645 795
1.026	011 147 360 775 797	1.043	018 284 308 426 531
1.02625	011 253 170 127 497	1.044	018 700 498 666 243
1.0265	011 358 953 706 611	1.045	019 116 290 447 073
1.02675	011 464 711 525 690	1.046	019 531 684 531 255
1.027	011 570 443 597 278	1.0475	020 154 031 638 333
1.02725	011 676 149 933 909	1.048	020 361 282 647 708

TABLE OF INTEREST RATIOS—(*Concluded*)

1 + *i*	Logarithm	1 + *i*	Logarithm
1.049	020 775 488 193 558	1.07	029 383 777 685 210
1.05	021 189 299 069 938	1.075	031 408 464 251 624
1.055	023 252 459 633 711	1.08	033 423 755 486 950
1.06	025 305 865 264 770	1.09	037 426 497 940 624
1.065	027 349 607 774 757	1.10	041 392 685 158 225

§ 360.

TABLE OF SUB-RECIPROCALS

Initial Figures	Sub-Reciprocal	Initial Figures	Sub-Reciprocal
10	90	35–36	27
11	83	37	26
12	76	38–39	25
13	71	40	24
14	66	41–42	23
15	62	43–44	22
16	58	45–46	21
17	55	47–49	20
18	52	50–51	19
19	50	52–54	18
20	47	55–57	17
21	45	58–61	16
22	43	62–65	15
23	41	66–70	14
24	40	71–75	13
25	38	76–82	12
26	37	83–89	11
27	35	90	10
28	34		
29	33		
30	32		
31	31		
32	30		
33	29		
34	28		

§ 361.

TABLE OF MULTIPLES

1	2	3	4	5	6	7	8	9
001	002	003	004	005	006	007	008	009
002	004	006	008	010	012	014	016	018
003	006	009	012	015	018	021	024	027
004	008	012	016	020	024	028	032	036
005	010	015	020	025	030	035	040	045
006	012	018	024	030	036	042	048	054
007	014	021	028	035	042	049	056	063
008	016	024	032	040	048	056	064	072
009	018	027	036	045	054	063	072	081
010	020	030	040	050	060	070	080	090
011	022	033	044	055	066	077	088	099
012	024	036	048	060	072	084	096	108
013	026	039	052	065	078	091	104	117
014	028	042	056	070	084	098	112	126
015	030	045	060	075	090	105	120	135
016	032	048	064	080	096	112	128	144
017	034	051	068	085	102	119	136	153
018	036	054	072	090	108	126	144	162
019	038	057	076	095	114	133	152	171
020	040	060	080	100	120	140	160	180
021	042	063	084	105	126	147	168	189
022	044	066	088	110	132	154	176	198
023	046	069	092	115	138	161	184	207
024	048	072	096	120	144	168	192	216
025	050	075	100	125	150	175	200	225
026	052	078	104	130	156	182	208	234
027	054	081	108	135	162	189	216	243
028	056	084	112	140	168	196	224	252
029	058	087	116	145	174	203	232	261
030	060	090	120	150	180	210	240	270
031	062	093	124	155	186	217	248	279
032	064	096	128	160	192	224	256	288
033	066	099	132	165	198	231	264	297
034	068	102	136	170	204	238	272	306

Table of Multiples—(*Continued*)

1	2	3	4	5	6	7	8	9
035	070	105	140	175	210	245	280	315
036	072	108	144	180	216	252	288	324
037	074	111	148	185	222	259	296	333
038	076	114	152	190	228	266	304	342
039	078	117	156	195	234	273	312	351
040	080	120	160	200	240	280	320	360
041	082	123	164	205	246	287	328	369
042	084	126	168	210	252	294	336	378
043	086	129	172	215	258	301	344	387
044	088	132	176	220	264	308	352	396
045	090	135	180	225	270	315	360	405
046	092	138	184	230	276	322	368	414
047	094	141	188	235	282	329	376	423
048	096	144	192	240	288	336	384	432
049	098	147	196	245	294	343	392	441
050	100	150	200	250	300	350	400	450
051	102	153	204	255	306	357	408	459
052	104	156	208	260	312	364	416	468
053	106	159	212	265	318	371	424	477
054	108	162	216	270	324	378	432	486
055	110	165	220	275	330	385	440	495
056	112	168	224	280	336	392	448	504
057	114	171	228	285	342	399	456	513
058	116	174	232	290	348	406	464	522
059	118	177	236	295	354	413	472	531
060	120	180	240	300	360	420	480	540
061	122	183	244	305	366	427	488	549
062	124	186	248	310	372	434	496	558
063	126	189	252	315	378	441	504	567
064	128	192	256	320	384	448	512	576
065	130	195	260	325	390	455	520	585
066	132	198	264	330	396	462	528	594
067	134	201	268	335	402	469	536	603
068	136	204	272	340	408	476	544	612
069	138	207	276	345	414	483	552	621

TABLE OF MULTIPLES—(*Concluded*)

1	2	3	4	5	6	7	8	9
070	140	210	280	350	420	490	560	630
071	142	213	284	355	426	497	568	639
072	144	216	288	360	432	504	576	648
073	146	219	292	365	438	511	584	657
074	148	222	296	370	444	518	592	666
075	150	225	300	375	450	525	600	675
076	152	228	304	380	456	532	608	684
077	154	231	308	385	462	539	616	693
078	156	234	312	390	468	546	624	702
079	158	237	316	395	474	553	632	711
080	160	240	320	400	480	560	640	720
081	162	243	324	405	486	567	648	729
082	164	246	328	410	492	574	656	738
083	166	249	332	415	498	581	664	747
084	168	252	336	420	504	588	672	756
085	170	255	340	425	510	595	680	765
086	172	258	344	430	516	602	688	774
087	174	261	348	435	522	609	696	783
088	176	264	352	440	528	616	704	792
089	178	267	356	445	534	623	712	801
090	180	270	360	450	540	630	720	810
091	182	273	364	455	546	637	728	819
092	184	276	368	460	552	644	736	828
093	186	279	372	465	558	651	744	837
094	188	282	376	470	564	658	752	846
095	190	285	380	475	570	665	760	855
096	192	288	384	480	576	672	768	864
097	194	291	388	485	582	679	776	873
098	196	294	392	490	588	686	784	882
099	198	297	396	495	594	693	792	891

Part IV—Tables

CHAPTER XXXI

EXPLANATION OF TABLES USED

§ 362. Object of the Tables

Any value shown in the following tables* might have been ascertained by the rules given in the text; but it is convenient and time saving to have at hand, already worked out, those results which are most frequently needed.

§ 363. Degree of Accuracy

The tables shown give each value to eight decimal places, while the ordinary tables extend only to five or six decimals. This allows accurate computations to be made on sums up to one million dollars, to the nearest cent—a degree of accuracy which will meet any ordinary requirements.

§ 364. Rates and Periods

The rates used in the tables* are as follows: 1%, 1¼%, 1½%, 1¾%, 2%, 2¼%, 2½%, 2¾%, 3%, 3½%, 4%, 4½%, 5%, and 6%. These are the rates most commonly used, since most investments are on a semi-annual basis. Rules for intermediate rates will be found in §§ 375, 376.

The periods given are from 1 to 50, inclusive, and also

* See Chapter XXXII.

every 5th period thereafter, viz.: 55, 60, 65, 70, 75, 80, 85, 90, 95, 100. Rules for obtaining the values for periods intervening above 50, and for extending above 100 periods, will be found in § 374.

In all the following tables of compound interest, the principal is considered to be $1; for any other principal the tabular result must be multiplied by the number of dollars in the principal.

§ 365. Tables Shown

Tables for obtaining the following results are shown in Chapter XXXII.

Table I—Amount
II—Present Worth
III—Amount of Annuity
IV—Present Worth of Annuity
V—Sinking Fund
VI—Reciprocals and Square Roots

§ 366. Annuities—When Payable

"Annuity" in these tables signifies the ordinary annuity where the payment is made at the end of each period. This kind of annuity is the one most used in investment calculations. Annuities paid in advance, like the premiums in life insurance, are sometimes called annuities due (§ 75). Their amounts and present worths may be derived from the tables of ordinary annuities.

§ 367. Table I—Amount

This gives the amount to which $1, invested now at the rate i, will have accumulated at the end of n periods. The rates (i) are at the top of the table and the numbers of periods (n) are on the left-hand margin. Each term up to 50 periods is $1 + i$ times the term above it; or is the term

below it divided by $1 + i$. Each term may also be considered as that power of $1 + i$, the ratio of increase, whose exponent is on the left. Thus in the column 3%, where $i = .03$, the value for 9 periods is the 9th power of 1.03, or $1.03^9 =$ 1.30477318. If this be multiplied by 1.03 it gives the 10th term, 1.34391638; if divided by 1.03 it gives the 8th term, 1.26677008.

§ 368. Compound Interest

To find the compound interest, subtract 1 from the amount. Thus the compound interest at 3% for 9 periods is .30477318; for 25 periods, 1.09377793. It is unnecessary, therefore, to give a separate table of the compound interest. All the other tables might be derived from Table I. The second line in each column is the ratio of increase.

§ 369. Table II—Present Worth

This gives the present worth of $1 payable n periods from now at the rate i; or the principal which invested now will at the end of n periods have accumulated to $1; or $1 discounted for n periods at i. It proceeds in exactly the contrary manner to Table I, diminishing instead of increasing, each term being divided by $1 + i$ to produce the succeeding one, or multiplied by $1 + i$ to produce the preceding.

Each term may be obtained independently from the corresponding term in Table I, the two terms being reciprocals of each other. If we represent any term in Table I by (I) and the corresponding term in Table II by (II), then we may say: $(\mathrm{II}) = 1 \div (\mathrm{I})$ and $(\mathrm{I}) = 1 \div (\mathrm{II})$; or more briefly with negative exponents, $(\mathrm{II}) = (\mathrm{I})^{-1}$ and $(\mathrm{I}) = (\mathrm{II})^{-1}$.

The second line in each column is the discount ratio or reciprocal of $1 + i$; and each term below is the nth power of

that number. Thus in the 3% column the discount ratio is 1.03^{-1} or .97087379; the present worth for 9 periods is .76641673, or the 9th power of .97087379, which may be expressed 1.03^{-9} or $1 \div 1.03^{9}$. Multiplied by 1.03, it gives 1.03^{-8}, or .78940923; divided by 1.03 it gives 1.03^{-10} or .74409391. Each of these multiplied by its correlative in Table I will give unity: (II) $\times$ (I) $= 1$.

All these relations should be verified by experiment until thoroughly understood.

The compound discount is obtained by subtracting the present worth from 1. The compound discount for 9 periods at 3% is $1 - .76641673 = .23358327$. Since this operation is easy, it is unnecessary to give a separate table of compound discounts.

§ 370. Table III—Amount of Annuity

This table gives the amount to which an ordinary annuity will accumulate; that is, if $1 be invested at the end of each period, the total investment will, after *n* periods, reach this amount.

It is formed from Table I by adding together the same number of terms as of the periods required. The three top lines of Table I give the third line of Table III; the fourth line of Table III is the sum of the first four lines of Table I; but for this purpose the line marked 0 in Table I must be counted in. Thus in the 2% column, $1 + 1.02 + 1.0404$ in Table I gives the value for 3 periods in Table III, 3.0604; for 4 periods it is $1 + 1.02 + 1.0404 + 1.061208 = 4.121608$; etc.

According to the principles laid down in § 60, we might have proceeded in this way: Take the amount of $1 for 3 periods from Table I (not the third but the fourth line), 1.061208; drop the 1, giving .061208; divide by .02, producing 3.0604.

But where the figures have been rounded, this procedure would leave two places indeterminate. For example, the amount of $1 for 20 periods is 1.48594740; the compound interest is .48594740; this multiplied by 100 and divided by 2, gives 24.297370. We have, therefore, cut down our result from 8 decimals to 6. But by addition of the first 20 lines of Table I, we get in Table III, 24.29736980; a gain in accuracy of two places.

It will be better, therefore, to reverse the process and test the accuracy of the table by dividing the result in Table III by 100 and multiplying by 2.

$$24.29736980 \times 2 \div 100 = .4859473960$$
$$.4859473960 + 1 = 1.4859473960 = 1.4859474||$$

This not only tests the accuracy of the table, but adds two places.

This suggests that for very accurate and extensive computations we may extend Table I to 10 figures, the last of which will be nearly accurate.

For most questions of investment 6 decimals of Tables III and IV will be ample.

When the amount of an annuity due is required, it is obtained as follows: subtract one from the number of periods and subtract one from the number of dollars. Thus we have at 2%:

Amount of ordinary annuity,	4 periods,	4.121608
Amount of annuity due,	3 periods,	3.121608

§ 371. Table IV—Present Worth of Annuity

This table gives the present worth of an annuity and is derived from Table II, precisely as III is derived from I. In the same way as before, Table II may be extended two places; but after multiplying by the rate, the result must be subtracted from 1.

The present worth of an annuity of 20 periods at 2% is, by Table IV, 16.35143334, which $\times .02 = .3270286668$; $1 - .3270286668 = .6729713332$. Table II gives .67297133, which is correct as far as it goes.

Table IV is the one used in bond valuations for ascertaining premiums and discounts which, as we have seen, are merely present worths of annuities consisting of the difference between the cash and income rates. Any ordinary premium or discount where the principal does not exceed one million dollars may safely be computed by using 6 or 7 figures of the decimals.

To transform Table IV of ordinary annuities into one of annuities due, add one to the number of periods and add one dollar to the value.

§ 372. Table V—Sinking Fund

This table gives sinking funds. It answers the question: What sum shall be invested at the end of each of n periods so that the sum-total with all accumulations shall amount to $1 at the end of n periods?

Each term of Table V is the reciprocal of the corresponding term in Table III. $(V) = (III)^{-1}$. Thus, to find the sinking fund necessary to provide a total of $1 in 9 periods, we divide 1 by the total to which an annuity of $1 would accumulate in 9 periods. At 3%, the latter would be:

$$10.15910613;\ 1 \div 10.15910613 = .0984338570$$

which is the sinking fund required, carried two places further than in Table V.

Another method of deriving the sinking fund would be to divide the single interest (.03) by the compound interest (.30477318) of $1 from Table I, which will be found to give the same result: $i \div I$.

§ 373. Rent of Annuity

To find what annuity has a present worth of \$1, we have only to add to the rate of interest the sum taken from Table V. This gives the rent of an annuity which \$1 will purchase, and it is, therefore, unnecessary to provide a table for that purpose. It might also be obtained to 8 places from Table II, dividing the single interest by the compound discount. It could also be derived by finding the reciprocal of the corresponding term in Table IV.

§ 374. Extension of Time

The tables go as far as 100 periods only, but Tables I and II may be extended to as many periods as desired by multiplication. The values for 148 periods might be obtained by multiplying together those for 100 and 48 periods respectively. Thus, at 1%, Table I, we have:

	100	2.70481383
	48	1.61222608
Using contracted multiplication,		2.70481383
		1.62288830
		2704814
		540963
		54096
		5410
		1623
		22
		4.36077141

The last figure is not quite accurate, but we could have made it more so by getting 10 figure values for 100 and for 48 periods from Table III.

100(170.48138294×.01=1.7048138294)+1=2.7048138294
48(61.22260777×.01= .6122260777)+1=1.6122260777

2.7048138294
1.6228882976
270481383
54096277
5409628
540963
162289
1893
189
19
Correct result to 10 figures, 4.3607713911

To extend Table III, IV, or V as to time, it is easiest to extend Table I or II and thence derive the value required.

§ 375. Subdivision of Rates

Although the rates given in these tables are those most frequently required, yet it often happens that intermediate rates occur, especially in bond computations. It might be supposed that these inter-rates could be obtained by "splitting the difference" into as many parts as necessary. But a trial will show that this gives only a rough approximation.

In Table I, for 10 periods at the rate 3%, the amount

is	1.34391638
and at 2½% it is	1.28008454
Midway between them is	1.31200046
but this is not the true value for 2¾%; it is	1.31165103
hence the error must be	.00034943

and the approximation holds good for only 3 decimals. But the correction can be very closely computed.

§ 376. Interpolation

Sometimes, in compound interest processes and also in mathematical problems, we have a series of terms, all formed by the same law, and based upon another series. A familiar illustration in mathematics is the formation of squares, for example:

Numbers,	1		2		3		4		5		6	etc.
Squares,	1		4		9		16		25		36	etc.
1st Differences,		3		5		7		9		11		etc.
2nd Differences,			2		2		2		2			etc.

When a series of terms such as that described above is written down in a table opposite to certain equi-distant numbers called arguments, intermediate terms corresponding to certain given arguments may be inserted by a process called interpolation, consisting of three steps:

(1) Differencing.

(2) Multiplication of each difference by a fraction dependent on the fractional distance at which the inter-term is to be located.

(3) Application of these corrections to the preceding term.

Differencing has already been treated to some extent in §§ 250 and 276. To interpolate in Table I, 10 periods, a value for 2¾%, we first set down the two values next greater and next less, opposite their arguments (3% and 2½%).

$$\left.\begin{array}{ll} 3\% & 1.34391638 \\ 2\frac{1}{2}\% & 1.28008454 \end{array}\right\}\text{decreasing terms}$$

or

$$\left.\begin{array}{ll} 2\frac{1}{2}\% & 1.28008454 \\ 3\% & 1.34391638 \end{array}\right\}\text{increasing terms}$$

The decreasing series has some advantages which make it preferable.

Continuing the column, use only equi-distant arguments, for 4 or more lines.

3%	1.34391638
2½%	1.28008454
2%	1.21899442
1½%	1.16054083
1%	1.10462213
½%	1.05114013
0%	0.00000000;

and proceed to difference,

		D_1	D_2	D_3
3%	1.34391638	.06383184	.00274172	.00010519 etc.
2½%	1.28008454	.06109012	.00263653	etc.
2%	1.21899442	.05845359	etc.	
1½%	1.16054083	etc.		
	etc.			

Let this process be carried out to the 6th difference and we have the following values, which are all that we need to consider:

D_1	.06383184
D_2	.00274172
D_3	.00010519
D_4	.00000355
D_5	.00000010
D_6	.00000001

From these differences any value corresponding to rates between 3% and 2½% may be determined. Each D will be multiplied by a certain fraction (F) according to the

fractional distance from 3% where the interpoland is to be located.

For the distance .5 (which means halfway), the F's are always as follows:

F_1	.5
F_2	.125
F_3	.0625
F_4	.0390625
F_5	.02734375
F_6	.0068359375

Multiplying each D by its corresponding F:

$D_1 \times F_1 = .06383184 \times .5$	$= .03191592$
$D_2 \times F_2 = .00274172 \times .125$	$= .00034271$
$D_3 \times F_3 = .00010519 \times .0625$	$= .00000658$
$D_4 \times F_4 = .00000355 \times .0390625$	$= .00000014$
$D_5 \times F_5 = .00000010 \times .02734375$, which is too small to affect the final figure.	
$D_6 \times F_6$, and following products are also negligible.	
Total correction,	.03226535
Subtract from value at 3%,	1.34391638
Interpolated value at 2¾%,	1.31165103

By using the above series of F's (.5, .125, .0625, .0390625, etc.), any interval may be bisected. But the interval may also be split into 5 parts as well as into 2. ⅕ = .2; therefore .2 would be F_1 for the first 5th, .4 would be F_1 for the second 5th; and .6 and .8 would be F_1 for the third and fourth intervals, respectively.

We will now give the proper F's for interpolating nine values, each at one-tenth interval.

F_1	F_2	F_3	F_4	F_5
.1	.045	.0285	.0206625	.01611675
.2	.08	.048	.03360	.025536
.3	.105	.0595	.0401625	.02972025
.4	.12	.064	.04160	.029952
.5	.125	.0625	.0390625	.02734375
.6	.12	.056	.03360	.022848
.7	.105	.0455	.0261625	.01726725
.8	.08	.032	.01760	.011264
.9	.045	.0165	.0086625	.00537075

To find the value corresponding to 2.60% in the same table: Since the interval is .50, ⅕ of the interval is .10, and the intermediate arguments would be 2.90% at .2 distance from .3; 2.80% at .4; 2.70% at .6; and 2.60% at .8. Therefore, we must use the F's of .8 as above, multiplying by them the same differences previously obtained.

.06383184 × .8	.05106547
.00274172 × .08	.00021934
.00010519 × .032	.00000337
.00000355 × .0176	.00000006
The remaining terms are negligible.	
Total,	.05128824
Subtract from value at 3%,	1.34391638
Interpolated value at 2.60%,	1.29262814

Had we chosen the increasing series in our differencing, there would have been this variation in the application of the corrections, that the first, third, fifth, seventh, and all odd-numbered corrections would have to be added to the preceding term and the even-numbered ones subtracted.

We should have differenced thus:

		D_1	D_2	D_3
2½	1.28008454	.06383184	.00285054	.00011260
3	1.34391638	.06668238	.00296314	etc.
3½	1.41059876	.06964552	etc.	
4	1.48024428	etc.		
	etc.			

The D's and their products would have figured thus, in the first example:

	2½% (now the basis)		1.28008454
.06383184 × .5		+	.03191592
			1.31200046
.00285054 × .125		—	35632
			1.31164414
.00011260 × .0625		+	704
			1.31165118
.00000388 × .0390625		—	15
	2¾%, as before,		1.31165103

The F's already given are generally sufficient for any practical purpose, but even if a very unusual fractional rate requires computation, the F's may always be worked out by the following formula:

F_1 is always the distance from the first value, expressed decimally.

Subtract F_1 from 1, multiply F_1 by the remainder and divide the product by 2; this gives F_2.

Subtract F_1 from 2, multiply F_2 by the remainder and divide the product by 3, giving F_3. And so on.

Observe that it is always the original F_1 which is subtracted from 1, 2, 3, etc., and that the divisor is always the number of the F sought.

This will be plainer in symbols.

$$F_2 = F_1 \times (1 - F_1) \div 2$$
$$F_3 = F_2 \times (2 - F_1) \div 3$$
$$F_4 = F_3 \times (3 - F_1) \div 4$$
$$F_5 = F_4 \times (4 - F_1) \div 5$$
etc.
$$F_n = F_{n-1} \times (n - 1 - F_1) \div n$$

The F's already given should be worked out for practice by these formulas.

As an example, we give the F's of .24.

$F_1 = .24$
$F_2 = .24 \quad \times 0.76 \div 2 = .0912$
$F_3 = .0912 \quad \times 1.76 \div 3 = .053504$
$F_4 = .053504 \quad \times 2.76 \div 4 = .03691776$
$F_5 = .03691776 \times 3.76 \div 5 = .0277621552$

Where the rates given in the tables are more than ½% apart, interpolation is not practically useful.

§ 377. Table VI—Reciprocals and Square Roots

This table gives the reciprocals and the square roots of 120 of the most necessary ratios of increase.

The ratios begin at ½ of 1%, and increase by 40ths of 1% to 3%; by 4ths of 1% to 7%; and by 1% to 10%.

The second column, composed of reciprocals, gives the present worth of \$1 payable one period from now, like the second line of Table II. It is used for the purpose of discounting by multiplication rather than by division, the former operation being much easier. Any reciprocal may

be tested by multiplying it by the ratio standing opposite, which will give as the result, unity.

The third column, composed of square roots, gives the equivalent effective ratio for a half-period. Thus for an obligation at 6% semi-annually the ratio of increase is 1.03. If a quarter of a year (a half-period) has elapsed, the amount, if scientifically treated, is not 1.015 as used in actual business, but 1.01488916. If the loaner were to receive 1.015 as the amount after three months and reinvest at the same rate, he would have, at the end of the half-yearly period, not 1.03 to which he is entitled, but 1.030225 ($= 1.015^2$). But if he receives 1.01488916 and reinvests for the other quarter at the same rate, he will have at the end of the half-year $1.01488916^2 = 1.03$.

In other words, if .03 is the rate for each period, the equivalent effective rate for a half-period is .01488916. To receive or pay 3% each half-year is exactly the same in effect as receiving or paying 1.488916% each quarter.

Intermediate values in the second and third columns may be readily found by interpolation, usually requiring only one F.

CHAPTER XXXII

TABLES OF COMPOUND INTEREST, PRESENT WORTH, ANNUITIES, SINKING FUNDS, AND OTHER COMPUTATIONS

§ 378 (a).

Table I—Part 1
Amount of $1 at Compound Interest

Periods	1%	1¼%	1½%	1¾%	2%	2¼%	2½%
0	1.	1.	1.	1.	1.	1.	1.
1	1.01	1.0125	1.015	1.0175	1.02	1.0225	1.025
2	1.0201	1.02515625	1.030225	1.03530625	1.0404	1.04550625	1.050625
3	1.030301	1.03797070	1.04567838	1.05342411	1.061208	1.06903014	1.07689063
4	1.04060401	1.05094534	1.06136355	1.07185903	1.08243216	1.09308332	1.10381289
5	1.05101005	1.06408215	1.07728400	1.09061656	1.10408080	1.11767769	1.13140821
6	1.06152015	1.07738318	1.09344326	1.10970235	1.12616242	1.14282544	1.15969342
7	1.07213535	1.09085047	1.10984491	1.12912215	1.14868567	1.16853901	1.18868575
8	1.08285671	1.10448610	1.12649259	1.14888178	1.17165938	1.19483114	1.21840290
9	1.09368527	1.11829218	1.14338998	1.16898721	1.19509257	1.22171484	1.24886297
10	1.10462213	1.13227083	1.16054083	1.18944449	1.21899442	1.24920343	1.28008454
11	1.11566835	1.14642422	1.17794894	1.21025977	1.24337431	1.27731050	1.31208666
12	1.12682503	1.16075452	1.19561817	1.23143931	1.26824179	1.30604999	1.34488882
13	1.13809328	1.17526395	1.21355244	1.25298950	1.29360663	1.33543611	1.37851104
14	1.14947421	1.18995475	1.23175573	1.27491682	1.31947876	1.36548343	1.41297382
15	1.16096896	1.20482918	1.25023207	1.29722786	1.34586834	1.39620680	1.44829817
16	1.17257864	1.21988955	1.26898555	1.31992935	1.37278571	1.42762146	1.48450562
17	1.18430443	1.23513817	1.28802033	1.34302811	1.40024142	1.45974294	1.52161826
18	1.19614748	1.25057739	1.30734064	1.36653111	1.42824625	1.49258716	1.55965872
19	1.20810895	1.26620961	1.32695075	1.39044540	1.45681117	1.52617037	1.59865019
20	1.22019004	1.28203723	1.34685501	1.41477820	1.48594740	1.56050920	1.63861644
21	1.23239194	1.29806270	1.36705783	1.43953681	1.51566634	1.59562066	1.67958185
22	1.24471586	1.31428848	1.38756370	1.46472871	1.54597967	1.63152212	1.72157140
23	1.25716302	1.33071709	1.40837715	1.49036146	1.57689926	1.66823137	1.76461068
24	1.26973465	1.34735105	1.42950281	1.51644279	1.60843725	1.70576653	1.80872595
25	1.28243200	1.36419294	1.45094535	1.54298054	1.64060599	1.74414632	1.85394410
26	1.29525631	1.38124535	1.47270953	1.56998269	1.67341811	1.78338962	1.90029270
27	1.30820888	1.39851092	1.49480018	1.59745739	1.70688648	1.82351588	1.94780002
28	1.32129097	1.41599230	1.51722218	1.62541290	1.74102421	1.86454499	1.99649502
29	1.33450388	1.43369221	1.53998051	1.65385762	1.77584469	1.90649725	2.04640739

30	1.34784892	1.45161336	1.56308022	1.68280013	1.81136158	1.94939344	2.09756758
31	1.36132740	1.46975853	1.58652642	1.71224913	1.84758882	1.99325479	2.15000677
32	1.37494068	1.48813051	1.61032432	1.74221349	1.88454059	2.03810303	2.20375694
33	1.38869009	1.50673214	1.63447918	1.77270223	1.92223140	2.08396034	2.25885086
34	1.40257699	1.52556629	1.65899637	1.80372452	1.96067603	2.13084945	2.31532213
35	1.41660276	1.54463587	1.68388132	1.83528970	1.99988955	2.17879356	2.37320519
36	1.43076878	1.56394382	1.70913954	1.86740727	2.03988734	2.22781642	2.43253532
37	1.44507647	1.58349312	1.73477663	1.90008689	2.08068509	2.27794229	2.49334870
38	1.45952724	1.60328678	1.76079828	1.93333841	2.12229879	2.32919599	2.55568242
39	1.47412251	1.62332787	1.78721025	1.96717184	2.16474477	2.38160290	2.61957448
40	1.48886373	1.64361946	1.81401841	2.00159734	2.20803966	2.43518897	2.68506384
41	1.50375237	1.66416471	1.84122868	2.03662530	2.25220046	2.48998072	2.75219043
42	1.51878989	1.68496677	1.86884712	2.07226624	2.29724447	2.54600528	2.82099520
43	1.53397779	1.70602885	1.89687982	2.10853090	2.34318936	2.60329040	2.89152008
44	1.54931757	1.72735421	1.92533302	2.14543019	2.39005314	2.66186444	2.96380808
45	1.56481075	1.74894614	1.95421301	2.18297522	2.43785421	2.72175639	3.03790328
46	1.58045885	1.77080797	1.98352621	2.22117728	2.48661129	2.78299590	3.11385086
47	1.59626344	1.79294306	2.01327910	2.26004789	2.53634351	2.84561331	3.19169713
48	1.61222608	1.81535485	2.04347829	2.29959872	2.58707039	2.90963961	3.27148956
49	1.62834834	1.83804679	2.07413046	2.33984170	2.63881179	3.97510650	3.35327680
50	1.64463182	1.86102237	2.10524242	2.38078893	2.69158803	3.04204640	3.43710872
55	1.72852457	1.98028070	2.26794398	2.59652785	2.97173067	3.40002740	3.88877303
60	1.81669670	2.10718135	2.44321978	2.83181628	3.28103079	3.80013479	4.39978975
65	1.90936649	2.24221407	2.63204158	3.08842574	3.62252311	4.24732588	4.97795826
70	2.00676337	2.38589997	2.83545629	3.36828827	3.99955822	4.74714140	5.63210286
75	2.10912847	2.53879358	3.05459171	3.67351098	4.41583546	5.30577405	6.37220743
80	2.21671522	2.70148494	3.29066279	4.00639192	4.87543916	5.93014530	7.20956782
85	2.32978997	2.87460191	3.54497838	4.36943740	5.38287878	6.62799112	8.15696424
90	2.44863267	3.05881260	3.81894851	4.76538080	5.94313313	7.40795782	9.22885633
95	2.57353755	3.25482789	4.11409214	5.19720324	6.56169920	8.27970921	10.44160385
100	2.70481383	3.46340427	4.43204565	5.66815594	7.24464612	9.25404630	11.81371635

§ 378 (b).

Table I—Part 2
Amount of $1 at Compound Interest

Periods	2¾%	3%	3½%	4%	4½%	5%	6%
0	1.	1.	1.	1.	1.	1.	1.
1	1.0275	1.03	1.035	1.04	1.045	1.05	1.06
2	1.05575625	1.0609	1.071225	1.0816	1.092025	1.1025	1.1236
3	1.08478955	1.092727	1.10871788	1.124864	1.14116613	1.157625	1.191016
4	1.11462126	1.12550881	1.14752300	1.16985856	1.19251860	1.21550625	1.26247696
5	1.14527334	1.15927407	1.18768631	1.21665290	1.24613194	1.27628156	1.33822558
6	1.17676836	1.19405230	1.22925533	1.26531902	1.30226012	1.34009564	1.41851911
7	1.20912949	1.22987387	1.27227926	1.31593178	1.36086183	1.40710042	1.50363026
8	1.24238055	1.26677008	1.31680904	1.36856905	1.42210061	1.47745544	1.59384807
9	1.27654602	1.30477318	1.36289735	1.42331181	1.48609514	1.55132822	1.68947896
10	1.31165103	1.34391638	1.41059876	1.48024428	1.55296942	1.62889463	1.79084770
11	1.34772144	1.38423387	1.45996972	1.53945406	1.62285305	1.71033936	1.89829856
12	1.38478378	1.42576089	1.51106866	1.60103222	1.69588143	1.79585633	2.01219647
13	1.42286533	1.46853371	1.56395606	1.66507351	1.77219610	1.88564914	2.13292826
14	1.46199413	1.51258972	1.61869452	1.73167645	1.85194492	1.97993160	2.26090396
15	1.50219896	1.55796742	1.67534883	1.80094351	1.93528244	2.07892818	2.39655819
16	1.54350944	1.60470644	1.73398604	1.87298125	2.02237015	2.18287459	2.54035168
17	1.58595595	1.65284763	1.79467555	1.94790050	2.11337681	2.29201832	2.69277279
18	1.62956973	1.70243306	1.85748920	2.02581652	2.20847877	2.40661923	2.85433915
19	1.67438290	1.75350605	1.92250132	2.10684918	2.30786031	2.52695020	3.02559950
20	1.72042843	1.80611123	1.98978886	2.19112314	2.41171402	2.65329771	3.20713547
21	1.76774021	1.86029457	2.05943147	2.27876807	2.52024116	2.78596259	3.39956360
22	1.81635307	1.91610341	2.13151158	2.36991879	2.63365201	2.92526072	3.60353742
23	1.86630278	1.97358651	2.20611448	2.46471554	2.75216635	3.07152376	3.81974966
24	1.91762610	2.03279411	2.28332849	2.56330416	2.87601383	3.22509994	4.04893464
25	1.97036082	2.09377793	2.36324498	2.66583633	3.00543446	3.38635494	4.29187072
26	2.02454575	2.15659127	2.44595856	2.77246978	3.14067901	3.55567269	4.54938296
27	2.08022075	2.22128901	2.53156711	2.88336858	3.28200956	3.73345632	4.82234594
28	2.13742682	2.28792768	2.62017196	2.99870332	3.42969999	3.92012914	5.11168670
29	2.19620606	2.35656551	2.71187798	3.11865145	3.58403649	4.11613560	5.41838790

30	2.25660173	2.42726247	2.80679370	3.24339751	3.74531813	4.32194238	5.74349117
31	2.31865828	2.50008035	2.90503148	3.37313341	3.91385745	4.53803949	6.08810064
32	2.38242138	2.57508276	3.00670759	3.50805875	4.08998104	4.76494147	6.45338668
33	2.44793797	2.65233524	3.11194235	3.64838110	4.27403018	5.00318854	6.84058988
34	2.51525626	2.73190530	3.22086033	3.79431634	4.46636154	5.25334797	7.25102528
35	2.58442581	2.81386245	3.33359045	3.94608899	4.66734781	5.51601537	7.68608679
36	2.65549752	2.89827833	3.45026611	4.10393255	4.87737846	5.79181614	8.14725200
37	2.72852370	2.98522668	3.57102543	4.26808986	5.09686049	6.08140694	8.63608712
38	2.80355810	3.07478348	3.69601132	4.43881345	5.32621921	6.38547729	9.15425235
39	2.88065595	3.16702698	3.82537171	4.61636599	5.56589908	6.70475115	9.70350749
40	2.95987399	3.26203779	3.95925972	4.8.102063	5.81636454	7.03998871	10.28571794
41	3.04127052	3.35989893	4.09783381	4.99306145	6.07810094	7.39198815	10.90286101
42	3.12490546	3.46069589	4.24125799	5.19278391	6.35161548	7.76158756	11.55703267
43	3.21084036	3.56451677	4.38970202	5.40049527	6.63743818	8.14966693	12.25045463
44	3.29913847	3.67145227	4.54334160	5.61651508	6.93612290	8.55715028	12.98548191
45	3.38986478	3.78159584	4.70235855	5.84117568	7.24824843	8.98500779	13.76461083
46	3.48308606	3.89504372	4.86694110	6.07482271	7.57441961	9.43425818	14.59048748
47	3.57887093	4.01189503	5.03728404	6.31781562	7.91526849	9.90597109	15.46591673
48	3.67728988	4.13225188	5.21358898	6.57052824	8.27145557	10.40126965	16.39387173
49	3.77841535	4.25621944	5.39606459	6.83334937	8.64367107	10.92133313	17.37750403
50	3.88232177	4.38390602	5.58492686	7.10668335	9.03263627	11.46739979	18.42015427
55	4.44631964	5.08214859	6.63314114	8.64636692	11.25630817	14.63563092	24.65032159
60	5.09225136	5.89160310	7.87809090	10.51962741	14.02740793	18.67918589	32.98769085
65	5.83201974	6.82998273	9.35670068	12.79873522	17.48070239	23.83990056	44.14497165
70	6.67925676	7.91782191	11.11282526	15.57161835	21.78413558	30.42642554	59.07593018
75	7.64957472	9.17892567	13.19855038	18.94525466	27.14699629	38.83268592	79.05692079
80	8.76085402	10.64089056	15.67573754	23.04979907	33.83009643	49.56144107	105.79599348
85	10.03357258	12.33570855	18.61785881	28.04360494	42.15845513	63.25435344	141.57890449
90	11.49118322	14.30046711	22.11217595	34.11933334	52.53710530	80.73036505	189.46451123
95	13.16054584	16.57816077	26.26232856	41.51138594	65.47079168	103.03467645	253.54625498
100	15.07242234	19.21863198	31.19140798	50.50494818	81.58851803	131.50125785	339.30208351

§ 379 (a).

TABLE II—PART 1
PRESENT WORTH OF $1 AT COMPOUND INTEREST

Periods	1%	1¼%	1½%	1¾%	2%	2¼%	2½%
0	1.	1.	1.	1.	1.	1.	1.
1	0.99009901	0.98765432	0.98522167	0.98280098	0.98039216	0.97799511	0.97560976
2	0.98029605	0.97546106	0.97066175	0.96589777	0.96116878	0.95647444	0.95181440
3	0.97059015	0.96341833	0.95631699	0.94928528	0.94232233	0.93542732	0.92859941
4	0.96098034	0.95152428	0.94218423	0.93295851	0.92384543	0.91484335	0.90595064
5	0.95146569	0.93977706	0.92826033	0.91691254	0.90573081	0.89471232	0.88385429
6	0.94204524	0.92817488	0.91454219	0.90114254	0.88797138	0.87502427	0.86229687
7	0.93271805	0.91671593	0.90102679	0.88564378	0.87056018	0.85576946	0.84126524
8	0.92348322	0.90539845	0.88771112	0.87041157	0.85349037	0.83693835	0.82074657
9	0.91433982	0.89422069	0.87459224	0.85544135	0.83675527	0.81852161	0.80072836
10	0.90528695	0.88318093	0.86166723	0.84072860	0.82034830	0.80051013	0.78119840
11	0.89632372	0.87227746	0.84893323	0.82626889	0.80426304	0.78289499	0.76214478
12	0.88744923	0.86150860	0.83638742	0.81205788	0.78849318	0.76566748	0.74355589
13	0.87866260	0.85087269	0.82402702	0.79809128	0.77303253	0.74881905	0.72542038
14	0.86996297	0.84036809	0.81184928	0.78436490	0.75787502	0.73234137	0.70772720
15	0.86134947	0.82999318	0.79985150	0.77087459	0.74301473	0.71622628	0.69046556
16	0.85282126	0.81974635	0.78803104	0.75761631	0.72844581	0.70046580	0.67362493
17	0.84437749	0.80962602	0.77638526	0.74458605	0.71416256	0.68505212	0.65719506
18	0.83601731	0.79963064	0.76491159	0.73177990	0.70015937	0.66997763	0.64116591
19	0.82773992	0.78975866	0.75360747	0.71919401	0.68643076	0.65523484	0.62552772
20	0.81954447	0.78000855	0.74247042	0.70682458	0.67297133	0.64081647	0.61027094
21	0.81143017	0.77037881	0.73149795	0.69466789	0.65977582	0.62671538	0.59538629
22	0.80339621	0.76086796	0.72068763	0.68272028	0.64683904	0.61292457	0.58086467
23	0.79544179	0.75147453	0.71003708	0.67097817	0.63415592	0.59943724	0.56669724
24	0.78756613	0.74219707	0.69954392	0.65943800	0.62172149	0.58624668	0.55287535
25	0.77976844	0.73303414	0.68920583	0.64809632	0.60953087	0.57334639	0.53939059
26	0.77204796	0.72398434	0.67902052	0.63694970	0.59757928	0.56072997	0.52623472
27	0.76440392	0.71504626	0.66898574	0.62599479	0.58586204	0.54839117	0.51339973
28	0.75683557	0.70621853	0.65909925	0.61522829	0.57437455	0.53632388	0.50087778
29	0.74934215	0.69749978	0.64935887	0.60464697	0.56311231	0.52452213	0.48866125

30	0.74192292	0.68838867	0.63976243	0.59424764	0.55207089	0.51298008	0.47674269
31	0.73457715	0.68038387	0.63030781	0.58402716	0.54124597	0.50169201	0.46511481
32	0.72730411	0.67198407	0.62099292	0.57398247	0.53063330	0.49065233	0.45377055
33	0.72010307	0.66368797	0.61181568	0.56411053	0.52022873	0.47985558	0.44270298
34	0.71297334	0.65549429	0.60277407	0.55440839	0.51002817	0.46929641	0.43190534
35	0.70591420	0.64740177	0.59386608	0.54487311	0.50002761	0.45896960	0.42137107
36	0.69892495	0.63940916	0.58508974	0.53550183	0.49022315	0.44887002	0.41109372
37	0.69200490	0.63151522	0.57644309	0.52629172	0.48061093	0.43899268	0.40106705
38	0.68515337	0.62371873	0.56792423	0.51724002	0.47118719	0.42933270	0.39128492
39	0.67836967	0.61601850	0.55953126	0.50834400	0.46194822	0.41988528	0.38174139
40	0.67165314	0.60841334	0.55126232	0.49960098	0.45289042	0.41064575	0.37243062
41	0.66500311	0.60090206	0.54311559	0.49100834	0.44401021	0.40160954	0.36334695
42	0.65841892	0.59348352	0.53508925	0.48256348	0.43530413	0.39277216	0.35448483
43	0.65189992	0.58615656	0.52718153	0.47426386	0.42676875	0.38412925	0.34583886
44	0.64544546	0.57892006	0.51939067	0.46610699	0.41840074	0.37567653	0.33740376
45	0.63905492	0.57177290	0.51171494	0.45809040	0.41019680	0.36740981	0.32917440
46	0.63272764	0.56471397	0.50415265	0.45021170	0.40215373	0.35932500	0.32114576
47	0.62646301	0.55774219	0.49670212	0.44246850	0.39426836	0.35141809	0.31331294
48	0.62026041	0.55085649	0.48936170	0.43485848	0.38653761	0.34368518	0.30567116
49	0.61411921	0.54405579	0.48212975	0.42737934	0.37895844	0.33612242	0.29821576
50	0.60803882	0.53733905	0.47500468	0.42002883	0.37152788	0.32872608	0.29094221
55	0.57852808	0.50497892	0.44092800	0.38512970	0.33650425	0.29411528	0.25715052
60	0.55044962	0.47456760	0.40929597	0.35313025	0.30478227	0.26314856	0.22728359
65	0.52373392	0.44598775	0.37993321	0.32378956	0.27605069	0.23544226	0.20088557
70	0.49831486	0.41912905	0.35267692	0.29688670	0.25002761	0.21065309	0.17755358
75	0.47412949	0.39388787	0.32737599	0.27221914	0.22645771	0.18847391	0.15693149
80	0.45111794	0.37016679	0.30389015	0.24960114	0.20510973	0.16862993	0.13870457
85	0.42922324	0.34787426	0.28208917	0.22886242	0.18577420	0.15087528	0.12259463
90	0.40839119	0.32692425	0.26185218	0.20984682	0.16826142	0.13498997	0.10835579
95	0.38857620	0.30723591	0.24306699	0.19241118	0.15239955	0.12077719	0.09577073
100	0.36971121	0.28873326	0.22562944	0.17642422	0.13803297	0.10806084	0.08464737

§ 379 (b).

Table II—Part 2
Present Worth of $1 at Compound Interest

Periods	2¾%	3%	3½%	4%	4½%	5%	6%
0	1.	1.	1.	1.	1.	1.	1.
1	0.97323601	0.97087379	0.96618357	0.96153846	0.95693780	0.95238095	0.94339623
2	0.94718833	0.94259591	0.93351070	0.92455621	0.91572995	0.90702948	0.88999644
3	0.92183779	0.91514166	0.90194271	0.88899636	0.87629660	0.86383760	0.83961928
4	0.89716573	0.88848705	0.87144223	0.85480419	0.83856134	0.82270247	0.79209366
5	0.87315400	0.86260878	0.84197317	0.82192711	0.80245105	0.78352617	0.74725817
6	0.84978491	0.83748426	0.81350064	0.79031453	0.76789574	0.74621540	0.70496054
7	0.82704128	0.81309151	0.78599096	0.75991781	0.73482846	0.71068133	0.66505711
8	0.80490635	0.78940923	0.75941156	0.73069021	0.70318513	0.67683936	0.62741237
9	0.78336385	0.76641673	0.73373097	0.70258674	0.67290443	0.64460892	0.59189846
10	0.76239791	0.74409391	0.70891881	0.67556417	0.64392768	0.61391325	0.55839478
11	0.74199310	0.72242128	0.68494571	0.64958093	0.61619874	0.58467929	0.52678753
12	0.72213440	0.70137988	0.66178330	0.62459705	0.58966386	0.55683742	0.49696936
13	0.70280720	0.68095134	0.63940415	0.60057409	0.56427164	0.53032135	0.46883902
14	0.68399728	0.66111781	0.61778179	0.57747508	0.53997286	0.50506795	0.44230096
15	0.66569078	0.64186195	0.59689062	0.55526450	0.51672044	0.48101710	0.41726506
16	0.64787424	0.62316694	0.57670591	0.53390818	0.49446932	0.45811152	0.39364628
17	0.63053454	0.60501645	0.55720378	0.51337325	0.47317639	0.43629669	0.37136442
18	0.61365892	0.58739461	0.53836114	0.49362812	0.45230037	0.41552065	0.35034379
19	0.59723496	0.57028603	0.52015569	0.47464242	0.43330179	0.39573396	0.33051301
20	0.58125057	0.55367575	0.50256588	0.45638695	0.41464286	0.37688948	0.31180473
21	0.56569398	0.53754928	0.48557090	0.43883360	0.39678743	0.35894236	0.29415540
22	0.55055375	0.52189250	0.46915063	0.42195539	0.37970089	0.34184987	0.27750510
23	0.53581874	0.50669175	0.45328563	0.40572633	0.36335013	0.32557131	0.26179726
24	0.52147809	0.49193374	0.43795713	0.39012147	0.34770347	0.31006791	0.24697855
25	0.50752126	0.47760557	0.42314699	0.37511680	0.33273060	0.29530277	0.23299863
26	0.49393796	0.46369473	0.40883767	0.36068923	0.31840248	0.28124073	0.21981003
27	0.48071821	0.45018906	0.39501224	0.34681657	0.30469137	0.26784832	0.20736795
28	0.46785227	0.43707675	0.38165434	0.33347747	0.29157069	0.25509364	0.19563014
29	0.45533068	0.42434636	0.36874815	0.32065141	0.27901502	0.24294632	0.18455674

30	0.44314421	0.41198676	0.35627841	0.30831867	0.26700002	0.23137745	0.17411013
31	0.43128391	0.39998715	0.34423035	0.29646026	0.25550241	0.22035947	0.16425484
32	0.41974103	0.38833703	0.33258971	0.28505794	0.24449991	0.20986617	0.15495740
33	0.40850708	0.37702625	0.32134271	0.27409417	0.23397121	0.19987254	0.14618622
34	0.39757380	0.36604490	0.31047605	0.26355209	0.22389589	0.19035480	0.13791153
35	0.38693314	0.35538340	0.29997686	0.25341547	0.21425444	0.18129029	0.13010522
36	0.37657727	0.34503243	0.28983272	0.24366872	0.20502817	0.17265741	0.12274077
37	0.36649856	0.33498294	0.28003161	0.23429685	0.19619921	0.16443563	0.11579318
38	0.35668959	0.32522615	0.27056194	0.22528543	0.18775044	0.15660536	0.10923885
39	0.34714316	0.31575355	0.26141250	0.21662061	0.17966549	0.14914797	0.10305552
40	0.33785222	0.30655684	0.25257247	0.20828904	0.17192870	0.14204568	0.09722219
41	0.32880995	0.29762800	0.24403137	0.20027793	0.16452507	0.13528160	0.09171905
42	0.32000968	0.28895922	0.23577910	0.19257493	0.15744026	0.12883962	0.08652740
43	0.31144495	0.28054294	0.22780590	0.18516820	0.15066054	0.12270440	0.08162962
44	0.30310944	0.27237178	0.22010231	0.17804635	0.14417276	0.11686133	0.07700908
45	0.29499702	0.26443862	0.21265924	0.17119841	0.13796437	0.11129651	0.07265007
46	0.28710172	0.25673653	0.20546787	0.16461386	0.13202332	0.10599668	0.06853781
47	0.27941773	0.24925876	0.19851968	0.15828256	0.12633810	0.10094921	0.06465831
48	0.27193940	0.24199880	0.19180645	0.15219476	0.12089771	0.09614211	0.06099840
49	0.26466122	0.23495029	0.18532024	0.14634112	0.11569158	0.09156391	0.05754566
50	0.25757783	0.22810708	0.17905337	0.14071262	0.11070965	0.08720373	0.05428836
55	0.22490511	0.19676717	0.15075814	0.11565551	0.08883907	0.06832640	0.04056742
60	0.19637679	0.16973309	0.12693431	0.09506040	0.07128901	0.05353552	0.03031434
65	0.17146718	0.14641325	0.10687528	0.07813272	0.05720594	0.04194648	0.02265264
70	0.14971726	0.12629736	0.08998612	0.06421940	0.04590497	0.03286617	0.01692737
75	0.13072622	0.10894521	0.07576590	0.05278367	0.03683649	0.02575150	0.01264911
80	0.11414412	0.09397710	0.06379285	0.04338433	0.02955948	0.02017698	0.00945215
85	0.09966540	0.08106547	0.05371187	0.03565875	0.02372003	0.01580919	0.00706320
90	0.08702324	0.06992779	0.04522395	0.02930890	0.01903417	0.01238691	0.00527803
95	0.07598469	0.06032032	0.03807735	0.02408978	0.01527399	0.00970547	0.00394405
100	0.06634634	0.05203284	0.03206011	0.01980004	0.01225663	0.00760449	0.00294723

§ 380 (a).

Table III—Part 1
Amount of Annuity of $1 at End of Each Period

Periods	1%	1¼%	1½%	1¾%	2%	2¼%	2½%
1	1.	1.	1.	1.	1.	1.	1.
2	2.01	2.0125	2.015	2.0175	2.02	2.0225	2.025
3	3.0301	3.03765625	3.045225	3.05280625	3.0604	3.06800625	3.075625
4	4.060401	4.07562695	4.09090338	4.10623036	4.121608	4.13703639	4.15251563
5	5.10100501	5.12657229	5.15226693	5.17808939	5.20404016	5.23011971	5.25632852
6	6.15201506	6.19065444	6.22955093	6.26870596	6.30812096	6.34779740	6.38773673
7	7.21353521	7.26803762	7.32299419	7.37840831	7.43428338	7.49062284	7.54743015
8	8.28567056	8.35888809	8.43283911	8.50753045	8.58296905	8.65916186	8.73611590
9	9.36852727	9.46337420	9.55933169	9.65641224	9.75462843	9.85399300	9.95451880
10	10.46221254	10.58166637	10.70272167	10.82539945	10.94972100	11.07570784	11.20338177
11	11.56683467	11.71393720	11.86326249	12.01484394	12.16871542	12.32491127	12.48346631
12	12.68250301	12.86036142	13.04121143	13.22510371	13.41208973	13.60222177	13.79555297
13	13.80932804	14.02111594	14.23682960	14.45654303	14.68033152	14.90827176	15.14044179
14	14.94742132	15.19637988	15.45038205	15.70953253	15.97393815	16.24370788	16.51895284
15	16.09689554	16.38633463	16.68213778	16.98444935	17.29341692	17.60919130	17.93192666
16	17.25786449	17.59116382	17.93236984	18.28167721	18.63928525	19.00539811	19.38022483
17	18.43044314	18.81105336	19.20135539	19.60160656	20.01207096	20.43301957	20.86473045
18	19.61474757	20.04619153	20.48937572	20.94463468	21.41231238	21.89276251	22.38634871
19	20.81089504	21.29676893	21.79671636	22.31116578	22.84055863	23.38534966	23.94600743
20	22.01900399	22.56297854	23.12366710	23.70161119	24.29736980	24.91152003	25.54465761
21	23.23919403	23.84501577	24.47052211	25.11638938	25.78331719	26.47202923	27.18327405
22	24.47158598	25.14307847	25.83757994	26.55592620	27.29898354	28.06764989	28.86285590
23	25.71630183	26.45736695	27.22514364	28.02065490	28.84496321	29.69917201	30.58442730
24	26.97346485	27.78808403	28.63352080	29.51101637	30.42186247	31.36740338	32.34903798
25	28.24319950	29.13543508	30.06302361	31.02745915	32.03029972	33.07316996	34.15776393
26	29.52563150	30.49962802	31.51396896	32.57043969	33.67090572	34.81731628	36.01170803
27	30.82088781	31.88087337	32.98667850	34.14042238	35.34432383	36.60070590	37.91200073
28	32.12909669	33.27938429	34.48147867	35.73787977	37.05121031	38.42422178	39.85980075
29	33.45038766	34.69537659	35.99870085	37.36329267	38.79223451	40.28876677	41.85629577

30	34.78489153	36.12906880	37.53868137	39.01715029	40.56807921	42.19526402	43.90270316
31	36.13274045	37.58068216	39.10176159	40.69995042	42.37944079	44.14465746	46.00027074
32	37.49406785	39.05044069	40.68828801	42.41219955	44.22702961	46.13791226	48.15027751
33	38.86900853	40.53857120	42.29861233	44.15441305	46.11157020	48.17601528	50.35403445
34	40.25769862	42.04530334	43.93309152	45.92711527	48.03380160	50.25997563	52.61288531
35	41.66027560	43.57086963	45.59208789	47.73083979	49.99447763	52.39082508	54.92820744
36	43.07687836	45.11550550	47.27596921	49.56612949	51.99436719	54.56961864	57.30141263
37	44.50764714	46.67944932	48.98510874	51.43353675	54.03425453	56.79743506	59.73394794
38	45.95272361	48.26294243	50.71988538	53.33362365	56.11493962	59.07537735	62.22729664
39	47.41225085	49.86622921	52.48068366	55.26696206	58.23723841	61.40457334	64.78297906
40	48.88637336	51.48955708	54.26789391	57.23413390	60.40198318	63.78617624	67.40255354
41	50.37523709	53.13317654	56.08191232	59.23573124	62.61002284	66.22136521	70.08761737
42	51.87898946	54.79734125	57.92314100	61.27235654	64.86222330	68.71134592	72.83980781
43	53.39777936	56.48230801	59.79198812	63.34462278	67.15946777	71.25735121	75.66080300
44	54.93175715	58.18833687	61.68886794	65.45315367	69.50265712	73.86064161	78.55232308
45	56.48107472	59.91569108	63.61420096	67.59858386	71.89271027	76.52250605	81.51613116
46	58.04588547	61.66463721	65.56841398	69.78155908	74.33056447	79.24426243	84.55403443
47	59.62634432	63.43544518	67.55194018	72.00273637	76.81717576	82.02725834	87.66788530
48	61.22260777	65.22838824	69.56521929	74.26278425	79.35351927	84.87287165	90.85958243
49	62.83483385	67.04374310	71.60869758	76.56238298	81.94058966	87.78251126	94.13107199
50	64.46318218	68.88178989	73.68282804	78.90222468	84.57940145	90.75761776	97.48434879
55	72.85245735	78.42245562	84.52959893	91.23016259	98.58653365	106.66788460	115.55092136
60	81.66966986	88.57450776	96.21465171	104.67521588	114.05153942	124.45043493	135.99158995
65	90.93664882	99.37712526	108.80277215	119.33861370	131.12615541	144.32559477	159.11833027
70	100.67633684	110.87199776	122.36375295	135.33075826	149.97791114	166.53961758	185.28411421
75	110.91284684	123.10348644	136.97278063	152.77205601	170.79177276	191.36773536	214.88829705
80	121,67152172	136.11879526	152.71085247	171.79382424	193.77195780	219.11756877	248.38271265
85	132.97899715	149.96815310	169.66522551	192.53927976	219.14393897	250.13293857	286.27856955
90	144.86326746	164.70500762	187.92990038	215.16461718	247.15665632	284.79812555	329.15425328
95	157.35375501	180.38623151	207.60614246	239.84018495	278.08495978	323.54263177	377.66415398
100	170.48138294	197.07234200	228.80304330	266.75176789	312.23230591	366.84650213	432.54865404

§ 380 (b).

Table III—Part 2
Amount of Annuity of $1 at End of Each Period

Period	2¾%	3%	3½%	4%	4½%	5%	6%
1	1.	1.	1.	1.	1.	1.	1.
2	2.0275	2.03	2.035	2.04	2.045	2.05	2.06
3	3.08325625	3.0909	3.106225	3.1216	3.137025	3.1525	3.1836
4	4.16804580	4.183627	4.21494288	4.246464	4.27819113	4.310125	4.374616
5	5.28266706	5.30913581	5.36246588	5.41632256	5.47070973	5.52563125	5.63709296
6	6.42794040	6.46840988	6.55015218	6.63297546	6.71689166	6.80191281	6.97531854
7	7.60470876	7.66246218	7.77940751	7.89829448	8.01915179	8.14200845	8.39383765
8	8.81383825	8.89233605	9.05168677	9.21422626	9.38001362	9.54910888	9.89746791
9	10.05621880	10.15910613	10.36849581	10.58279531	10.80211423	11.02656432	11.49131598
10	11.33276482	11.46387931	11.73139316	12.00610712	12.28820937	12.57789254	13.18079494
11	12.64441585	12.80779569	13.14199192	13.48635141	13.84117879	14.20678716	14.97164264
12	13.99213729	14.19202956	14.60196164	15.02580546	15.46403184	15.91712652	16.86994120
13	15.37692107	15.61779045	16.11303030	16.62683768	17.15991327	17.71298285	18.88213767
14	16.79978639	17.08632416	17.67698636	18.29191119	18.93210937	19.59863199	21.01506593
15	18.26178052	18.59891389	19.29568088	20.02358764	20.78405429	21.57856359	23.27596988
16	19.76397948	20.15688130	20.97102971	21.82453114	22.71933673	23.65749177	25.67252808
17	21.30748892	21.76158774	22.70501575	23.69751239	24.74170689	25.84036636	28.21287976
18	22.89344487	23.41443537	24.49969130	25.64541288	26.85508370	28.13238467	30.90565255
19	24.52301460	25.11686844	26.35718050	27.67122940	29.06356246	30.53900391	33.75999170
20	26.19739750	26.87037449	28.27968181	29.77807858	31.37142277	33.06595410	36.78559120
21	27.91782593	28.67648572	30.26947068	31.96920172	33.78313680	35.71925181	39.99272668
22	29.68556615	30.53678030	32.32890215	34.24796979	36.30337795	38.50521440	43.39229028
23	31.50191921	32.45288370	34.46041373	36.61788858	38.93702996	41.43047512	46.99582769
24	33.36822199	34.42647022	36.66652821	39.08260412	41.68919631	44.50199887	50.81557735
25	35.28584810	36.45926432	38.94985669	41.64590829	44.56521015	47.72709882	54.86451200
26	37.25620892	38.55304225	41.31310168	44.31174462	47.57064460	51.11345376	59.15638272
27	39.28075467	40.70963352	43.75906024	47.08421440	50.71132361	54.66912645	63.70576568
28	41.36097542	42.93092252	46.29062734	49.96758298	53.99333317	58.40258277	68.52811162
29	43.49840224	45.21885020	48.91079930	52.96628630	57.42303316	62.32271191	73.63979832

30	45.69460830	47.57541571	51.62267728	56.08493775	61.00706966	66.43884750	79.05818622
31	47.95121003	50.00267818	54.42947098	59.32833526	64.75238779	70.76078988	84.80167739
32	50.26986831	52.50275852	57.33450247	62.70146867	68.66624524	75.29882937	90.88977803
33	52.65228969	55.07784128	60.34121005	66.20952742	72.75622628	80.06377084	97.34316471
34	55.10022765	57.73017652	63.45315240	69.85790851	77.03025646	85.06695938	104.18375460
35	57.61548391	60.46208181	66.67401274	73.65222486	81.49661800	90.32030735	111.43477987
36	60.19990972	63.27594427	70.00760318	77.59831385	86.16396581	95.83632272	119.12086666
37	62.85540724	66.17422259	73.45786930	81.70224640	91.04134427	101.62813886	127.26811866
38	65.58393094	69.15944927	77.02889472	85.97033626	96.13820476	107.70954580	135.90420578
39	68.38748904	72.23423275	80.72490604	90.40914971	101.46442398	114.09502309	145.05845813
40	71.26814499	75.40125973	84.55027775	95.02551570	107.03032306	120.79977424	154.76196562
41	74.22801898	78.66329753	88.50953747	99.82653633	112.84668760	127.83976295	165.04768356
42	77.26928950	82.02319645	92.60737128	104.81959778	118.92478854	135.23175110	175.95054457
43	80.39419496	85.48389234	96.84862928	110.01238169	125.27640402	142.99333866	187.50757724
44	83.60503532	89.04840911	101.23833130	115.41287696	131.91384220	151.14300559	199.75803188
45	86.90417379	92.71986139	105.78167290	121.02939204	138.84996510	159.70015587	212.74351379
46	90.29403857	96.50145723	110.48403145	126.87056772	146.09821353	168.68516366	226.50812462
47	93.77712463	100.39650095	115.35097255	132.94539043	153.67263314	178.11942185	241.09861210
48	97.35599556	104.40839598	120.38825659	139.26320604	161.58790163	188.02539294	256.56452882
49	101.03328544	108.54064785	125.60184557	145.83373429	169.85935720	198.42666259	272.95840055
50	104.81170079	112.79686729	130.99791016	152.66708366	178.50302828	209.34799572	290.33590458
55	125.32071411	136.07161972	160.94688984	191.15917299	227.91795938	272.71261833	394.17202657
60	148.80914038	163.05343680	196.51688288	237.99068520	289.49795398	353.58371788	533.12818089
65	175.70980889	194.33275782	238.76287650	294.96838045	366.23783096	456.79801118	719.08286076
70	206.51842746	230.59406374	288.93786459	364.29045876	461.86967955	588.52851071	967.93216965
75	241.80271709	272.63085559	348.53001083	448.63136652	581.04436193	756.65371848	1300.94867977
80	282.21287345	321.36301855	419.30678685	551.24497675	729.55769854	971.22882134	1746.59989137
85	328.49354837	377.85695165	503.36739448	676.09012345	914.63233612	1245.08706889	2342.98174142
90	381.49757170	443.34890365	603.20502701	827.98333354	1145.26900659	1594.60730098	3141.07518718
95	442.20166674	519.27202569	721.78081595	1012.78464845	1432.68425949	2040.69352892	4209.10424961
100	511.72444867	607.28773270	862.61165666	1237.62370461	1790.85595627	2610.02515693	5638.36805857

§ 381 (a).

Table IV—Part 1

Present Worth of Annuity of $1 at End of Each Period

Periods	1%	1¼%	1½%	1¾%	2%	2¼%	2½%
1	0.99009901	0.98765432	0.98522167	0.98280098	0.98039216	0.97799511	0.97560976
2	1.97039506	1.96311538	1.95588342	1.94869875	1.94156094	1.93446955	1.92742415
3	2.94098521	2.92653371	2.91220042	2.89798403	2.88388327	2.86989687	2.85602356
4	3.90196555	3.87805798	3.85438465	3.83094254	3.80772870	3.78474021	3.76197421
5	4.85343124	4.81783504	4.78264497	4.74785508	4.71345951	4.67945253	4.64582850
6	5.79547647	5.74600992	5.69718717	5.64899762	5.60143089	5.55447680	5.50812536
7	6.72819453	6.66272585	6.59821396	6.53464139	6.47199107	6.41024626	6.34939060
8	7.65167775	7.56812429	7.48592508	7.40505297	7.32548144	7.24718461	7.17013717
9	8.56601758	8.46234498	8.36051732	8.26049432	8.16223671	8.06570622	7.97086553
10	9.47130453	9.34552591	9.22218455	9.10122291	8.98258501	8.86621635	8.75206393
11	10.36762825	10.21780337	10.07111779	9.92749181	9.78684805	9.64911134	9.51420871
12	11.25507747	11.07931197	10.90750521	10.73954969	10.57534122	10.41477882	10.25776460
13	12.13374007	11.93018466	11.73153222	11.53764097	11.34837375	11.16359787	10.98318497
14	13.00370304	12.77055275	12.54338150	12.32200587	12.10624877	11.89593924	11.69091217
15	13.86505252	13.60054592	13.34323301	13.09288046	12.84926350	12.61216551	12.38137773
16	14.71787378	14.42029227	14.13126405	13.85049677	13.57770931	13.31263131	13.05500266
17	15.56225127	15.22991829	14.90764931	14.59508282	14.29187188	13.99768343	13.71219772
18	16.39826858	16.02954893	15.67256089	15.32686272	14.99203125	14.66766106	14.35336363
19	17.22600850	16.81930759	16.42616837	16.04605673	15.67846201	15.32289590	14.97889134
20	18.04555297	17.59931613	17.16863879	16.75288130	16.35143334	15.96371237	15.58916229
21	18.85698313	18.36969495	17.90013673	17.44754919	17.01120916	16.59042775	16.18454857
22	19.66037934	19.13056291	18.62082437	18.13026948	17.65804820	17.20335232	16.76541324
23	20.45582113	19.88203744	19.33086145	18.80124764	18.29220412	17.80278955	17.33211048
24	21.24338726	20.62423451	20.03040537	19.46068565	18.91392560	18.38903624	17.88498583
25	22.02315570	21.35726865	20.71961120	20.10878196	19.52345647	18.96238263	18.42437642
26	22.79520366	22.08125299	21.39863172	20.74573166	20.12103576	19.52311260	18.95061114
27	23.55960759	22.79629925	22.06761746	21.37172644	20.70689780	20.07150376	19.46401087
28	24.31644316	23.50251778	22.72671671	21.98695474	21.28127236	20.60782764	19.96488866
29	25.06578530	24.20001756	23.37607558	22.59160171	21.84438466	21.13234977	20.45354991

30	25.80770822	24.88890623	24.01583801	23.18584934	22.39645555	21.64532985	20.93029259
31	26.54228537	25.56929010	24.64614582	23.76987650	22.93770152	22.14702186	21.39540741
32	27.26958947	26.24127418	25.26713874	24.34385897	23.46833482	22.63767419	21.84917796
33	27.98969255	26.90496215	25.87895442	24.90796951	23.98856355	23.11752977	22.29188094
34	28.70266589	27.56045644	26.48172849	25.46237789	24.49859172	23.58682618	22.72378628
35	29.40858009	28.20785822	27.07559458	26.00725100	24.99861933	24.04579577	23.14515734
36	30.10750504	28.84726737	27.66068431	26.54275283	25.48884248	24.49466579	23.55625107
37	30.79950994	29.47878259	28.23712740	27.06904455	25.96945341	24.93365848	23.95731812
38	31.48466330	30.10250133	28.80505163	27.58628457	26.44064060	25.36299118	24.34860304
39	32.16303298	30.71851983	29.36458288	28.09462857	26.90258883	25.78287646	24.73034443
40	32.83468611	31.32693316	29.91584520	28.59422955	27.35547924	26.19352221	25.10277505
41	33.49968922	31.92783522	30.45896079	29.08523789	27.79948945	26.59513174	25.46612200
42	34.15810814	32.52131874	30.99405004	29.56780136	28.23479358	26.98790390	25.82060683
43	34.81000806	33.10747530	31.52123157	30.04206522	28.66156233	27.37203316	26.16644569
44	35.45545352	33.68639536	32.04062223	30.50817221	29.07996307	27.74770969	26.50384945
45	36.09450844	34.25816825	32.55233718	30.96626261	29.49015987	28.11511950	26.83302386
46	36.72723608	34.82288222	33.05648983	31.41647431	29.89231360	28.47444450	27.15416962
47	37.35369909	35.38062442	33.55319195	31.85894281	30.28658196	28.82586259	27.46748255
48	37.97395949	35.93148091	34.04255365	32.29380129	30.67311957	29.16954777	27.77315371
49	38.58807871	36.47553670	34.52468339	32.72118063	31.05207801	29.50567019	28.07136947
50	39.19611753	37.01287574	34.99968807	33.14120946	31.42360589	29.83439627	28.36231168
55	42.14719216	39.60168667	37.27146681	35.13544550	33.17478752	31.37265438	29.71397928
60	44.95503841	42.03459179	39.38026889	36.96398552	34.76088668	32.74895285	30.90865649
65	47.62660777	44.32098022	41.33778618	38.64059678	36.19746555	33.98034405	31.96457705
70	50.16851435	46.46967562	43.15487183	40.17790267	37.49861929	35.08208492	32.89785698
75	52.58705124	48.48897027	44.84160034	41.58747771	38.67711433	36.06782605	33.72274044
80	54.88820611	50.38665706	46.40732349	42.87993474	39.74451359	36.94978079	34.45181722
85	57.07767600	52.17005958	47.86072218	44.06500479	40.71128999	37.73887655	35.09621486
90	59.16088148	53.84606035	49.20985452	45.15161037	41.58692916	38.44489025	35.66576848
95	61.14298002	55.42112744	50.46220054	46.14793265	42.38002254	39.07656940	36.16917089
100	63.02887877	56.90133936	51.62470367	47.06147304	43.09835164	39.64174052	36.61410526

§ 381 (b). TABLE IV—PART 2

PRESENT WORTH OF ANNUITY OF $1 AT END OF EACH PERIOD

Periods	2¾%	3%	3½%	4%	4½%	5%	6%
1	0.97323601	0.97087379	0.96618357	0.96153846	0.95693780	0.95238095	0.94339623
2	1.92042434	1.91346970	1.89969428	1,88609467	1.87266775	1.85941043	1.83339267
3	2.84226213	2.82861135	2.80163698	2.77509103	2.74896435	2.72324803	2.67301195
4	3.73942787	3.71709840	3.67307921	3.62989522	3.58752570	3.54595050	3.46510561
5	4.61258186	4.57970719	4.51505238	4.45182233	4.38997674	4.32947667	4.21236379
6	5.46236678	5.41719144	5.32855302	5.24213686	5.15787248	5.07569206	4.91732433
7	6.28940806	6.23028296	6.11454398	6.00205467	5.89270094	5.78637340	5.58238144
8	7.09431441	7.01969219	6.87395554	6.73274487	6.59588607	6.46321276	6.20979381
9	7.87767826	7.78610892	7.60768651	7.43533161	7.26879050	7.10782168	6.80169227
10	8.64007616	8.53020284	8.31660532	8.11089578	7.91271818	7.72173493	7.36008705
11	9.38206926	9.25262411	9.00155104	8.76047671	8.52891692	8.30641422	7.88687458
12	10.10420366	9.95400399	9.66333433	9.38507376	9.11858078	8.86325164	8.38384394
13	10.80701086	10.63495533	10.30273849	9.98564785	9.68285242	9.39357299	8.85268296
14	11.49100814	11.29607314	10.92052028	10.56312293	10.22282528	9.89864094	9.29498393
15	12.15669892	11.93793509	11.51741090	11.11838743	10.73954573	10.37965804	9.71224899
16	12.80457315	12.56110203	12.09411681	11.65229561	11.23401505	10.83776956	10.10589527
17	13.43510769	13.16611847	12.65132059	12.16566885	11.70719143	11.27406625	10.47725969
18	14.04876661	13.75351308	13.18968173	12.65929697	12.15999180	11.68958690	10.82760348
19	14.64600157	14.32379911	13.70983742	13.13393940	12.59329359	12.08532086	11.15811649
20	15.22725213	14.87747486	14.21240330	13.59032634	13.00793645	12.46221034	11.46992122
21	15.79294612	15.41502414	14.69797420	14.02915995	13.40472388	12.82115271	11.76407662
22	16.34349987	15.93691664	15.16712484	14.45111533	13.78442476	13.16300258	12.04158172
23	16.87931861	16.44360839	15.62041047	14.85684167	14.14777489	13.48857388	12.30337898
24	17.40079670	16.93554212	16.05836760	15.24696314	14.49547837	13.79864179	12.55035753
25	17.90831795	17.41314769	16.48151459	15.62207994	14.82820896	14.09394457	12.78335616
26	18.40225592	17.87684242	16.89035226	15.98276918	15.14661145	14.37518530	13.00316619
27	18.88297413	18.32703147	17.28536451	16.32958575	15.45130282	14.64303362	13.21053414
28	19.35082640	18.76410823	17.66701885	16.66306322	15.74287351	14.89812726	13.40616428
29	19.80615708	19.18845459	18.03576700	16.98371463	16.02188853	15.14107358	13.59072102

30	20.24930130	19.60044135	18.39204541	17.29203330	16.28888854	15.37245103	13.76483115
31	20.68058520	20.00042849	18.73627576	17.58849356	16.54439095	15.59281050	13.92908599
32	21.10032623	20.38876553	19.06886547	17.87355150	16.78889086	15.80267667	14.08404339
33	21.50883332	20.76579178	19.39020818	18.14764567	17.02286207	16.00254921	14.23022961
34	21.90640712	21.13183668	19.70068423	18.41119776	17.24675796	16.19290401	14.36814114
35	22.29334026	21.48722007	20.00066110	18.66461323	17.46101240	16.37419429	14.49824636
36	22.66991753	21.83225250	20.29049381	18.90828195	17.66604058	16.54685171	14.62098713
37	23.03641609	22.16723544	20.57052542	19.14257880	17.86223979	16.71128734	14.73678031
38	23.39310568	22.49246159	20.84108736	19.36786423	18.04999023	16.86789271	14.84601916
39	23.74024884	22.80821513	21.10249987	19.58448484	18.22965572	17.01704067	14.94907468
40	24.07810106	23.11477197	21.35507234	19.79277388	18.40158442	17.15908635	15.04629687
41	24.40691101	23.41239997	21.59910371	19.99305181	18.56610949	17.29436796	15.13801592
42	24.72692069	23.70135920	21.83488281	20.18562674	18.72354975	17.42320758	15.22454332
43	25.03836563	23.98190213	22.06268870	20.37079494	18.87421029	17.54591198	15.30617294
44	25.34147507	24.25427392	22.28279102	20.54884129	19.01838305	17.66277331	15.38318202
45	25.63647209	24.51871254	22.49545026	20.72003970	19.15634742	17.77406982	15.45583209
46	25.92357381	24.77544907	22.70091813	20.88465356	19.28837074	17.88006650	15.52436990
47	26.20299154	25.02470783	22.89943780	21.04293612	19.41470884	17.98101571	15.58902821
48	26.47493094	25.26670664	23.09124425	21.19513088	19.53560654	18.07715782	15.65002661
49	26.73959215	25.50165693	23.27656450	21.34147200	19.65129813	18.16872173	15.70757227
50	26.99716998	25.72976401	23.45561787	21.48218462	19.76200778	18.25592546	15.76186064
55	28.18526879	26.77442764	24.26405323	22.10861218	20.24802057	18.63347196	15.99054297
60	29.22266201	27.67556367	24.94473412	22.62348997	20.63802204	18.92928952	16.16142771
65	30.12846605	28.45289152	25.51784916	23.04668199	20.95097913	19.16107033	16.28912272
70	30.91937247	29.12342135	26.00039664	23.39451498	21.20211187	19.34267665	16.38454387
75	31.60995558	29.70182628	26.40668868	23.68040834	21.40363360	19.48496995	16.45584810
80	32.21294098	30.20076345	26.74877567	23.91539185	21.56534493	19.59646048	16.50913077
85	32.73944009	30.63115103	27.03680373	24.10853116	21.69511035	19.68381623	16.54894668
90	33.19915489	31.00240714	27.27931564	24.26727759	21.79924075	19.75226174	16.57869944
95	33.60055671	31.32265592	27.48350415	24.39775559	21.88280030	19.80589059	16.60093244
100	33.95104232	31.59890534	27.65542540	24.50499900	21.94985274	19.84791020	16.61754623

§ 382 (a).

Table V—Part 1

Sinking Fund or Annuity Which, Invested at the End of Each Period, Will Amount to $1

Periods	1%	1¼%	1½%	1¾%	2%	2¼%	2½%
1	1.	1.	1.	1.	1.	1.	1.
2	0.49751244	0.49689441	0.49627792	0.49566295	0.49504950	0.49443758	0.49382716
3	0.33002211	0.32920117	0.32838296	0.32756746	0.32675467	0.32594458	0.32513717
4	0.24628109	0.24536102	0.24444478	0.24353237	0.24262375	0.24171893	0.24081788
5	0.19603980	0.19506211	0.19408932	0.19312142	0.19215839	0.19120021	0.19024686
6	0.16254837	0.16153381	0.16052521	0.15952256	0.15852581	0.15753496	0.15654997
7	0.13862828	0.13758872	0.13655616	0.13553059	0.13451196	0.13350025	0.13249543
8	0.12069029	0.11963314	0.11858402	0.11754292	0.11650980	0.11548462	0.11446735
9	0.10674036	0.10567055	0.10460982	0.10355813	0.10251544	0.10148170	0.10045689
10	0.09558208	0.09450307	0.09343418	0.09237534	0.09132653	0.09028768	0.08925876
11	0.08645408	0.08536839	0.08429384	0.08323038	0.08217794	0.08113649	0.08010596
12	0.07884879	0.07775831	0.07667999	0.07561377	0.07455960	0.07351740	0.07248713
13	0.07241482	0.07132100	0.07024036	0.06917283	0.06811835	0.06707686	0.06604827
14	0.06690117	0.06580515	0.06472332	0.06365562	0.06260197	0.06156230	0.06053653
15	0.06212378	0.06102646	0.05994436	0.05887739	0.05782547	0.05678852	0.05576646
16	0.05794460	0.05684672	0.05576508	0.05469958	0.05365013	0.05261663	0.05159899
17	0.05425806	0.05316023	0.05207966	0.05101623	0.04996984	0.04894039	0.04792777
18	0.05098205	0.04988479	0.04880578	0.04774492	0.04670210	0.04567720	0.04467008
19	0.04805175	0.04695548	0.04587847	0.04482061	0.04378177	0.04276182	0.04176062
20	0.04541531	0.04432039	0.04324574	0.04219122	0.04115672	0.04014207	0.03914713
21	0.04303075	0.04193748	0.04086550	0.03981464	0.03878477	0.03777572	0.03678733
22	0.04086372	0.03977238	0.03870331	0.03765638	0.03663140	0.03562821	0.03464661
23	0.03888584	0.03779666	0.03673075	0.03568796	0.03466810	0.03367097	0.03269638
24	0.03707347	0.03598665	0.03492410	0.03388565	0.03287110	0.03188023	0.03091282
25	0.03540675	0.03432247	0.03326345	0.03222952	0.03122044	0.03023599	0.02927592
26	0.03386888	0.03278729	0.03173196	0.03070269	0.02969923	0.02872134	0.02776875
27	0.03244553	0.03136677	0.03031527	0.02929079	0.02829309	0.02732188	0.02637687
28	0.03112444	0.03004863	0.02900108	0.02798151	0.02698967	0.02602525	0.02508793
29	0.02980502	0.02882228	0.02777878	0.02676424	0.02577836	0.02482081	0.02389127

30	0.02874811	0.02767854	0.02663919	0.02562975	0.02464992	0.02369934	0.02277764
31	0.02767573	0.02660942	0.02557430	0.02457005	0.02359635	0.02265280	0.02173900
32	0.02667089	0.02560791	0.02457710	0.02357812	0.02261061	0.02167415	0.02076831
33	0.02572744	0.02466786	0.02364144	0.02264779	0.02168653	0.02075722	0.01985938
34	0.02483997	0.02378387	0.02276189	0.02177363	0.02081867	0.01989655	0.01900675
35	0.02400368	0.02295111	0.02193363	0.02095082	0.02000221	0.01908731	0.01820558
36	0.02321431	0.02216533	0.02115240	0.02017507	0.01923285	0.01832522	0.01745158
37	0.02246805	0.02142270	0.02041437	0.01944257	0.01850678	0.01760643	0.01674090
38	0.02176150	0.02071983	0.01971613	0.01874990	0.01782057	0.01692753	0.01607012
39	0.02109160	0.02005365	0.01905463	0.01809399	0.01717114	0.01628543	0.01543615
40	0.02045560	0.01942141	0.01842710	0.01747209	0.01655575	0.01567738	0.01483623
41	0.01985102	0.01882063	0.01783106	0.01688170	0.01597188	0.01510087	0.01426786
42	0.01927563	0.01824906	0.01726426	0.01632057	0.01541729	0.01455364	0.01372876
43	0.01872737	0.01770466	0.01672465	0.01578666	0.01488993	0.01403364	0.01321688
44	0.01820441	0.01718557	0.01621038	0.01527810	0.01438794	0.01353901	0.01273037
45	0.01770505	0.01669012	0.01571976	0.01479321	0.01390962	0.01306805	0.01226752
46	0.01722775	0.01621675	0.01525125	0.01433043	0.01345342	0.01261921	0.01182676
47	0.01677111	0.01576406	0.01480342	0.01388836	0.01301792	0.01219107	0.01140669
48	0.01633384	0.01533075	0.01437500	0.01346569	0.01260184	0.01178233	0.01100599
49	0.01591474	0.01491563	0.01396478	0.01306124	0.01220396	0.01139179	0.01062348
50	0.01551273	0.01451763	0.01357168	0.01267391	0.01182321	0.01101836	0.01025806
55	0.01372637	0.01275145	0.01183018	0.01096129	0.01014337	0.00937489	0.00865419
60	0.01224445	0.01128993	0.01039343	0.00955336	0.00876797	0.00803533	0.00735340
65	0.01099667	0.01006268	0.00919094	0.00837952	0.00762624	0.00692878	0.00628463
70	0.00993282	0.00901941	0.00817235	0.00738930	0.00666765	0.00600458	0.00539712
75	0.00901609	0.00812325	0.00730072	0.00654570	0.00585508	0.00522554	0.00465358
80	0.00821885	0.00734652	0.00654832	0.00582093	0.00516071	0.00456376	0.00402605
85	0.00751998	0.00666808	0.00589396	0.00519375	0.00456321	0.00399787	0.00349310
90	0.00690306	0.00607146	0.00532113	0.00464760	0.00404602	0.00351126	0.00303809
95	0.00635511	0.00554366	0.00481681	0.00416944	0.00359602	0.00309078	0.00264786
100	0.00586574	0.00507428	0.00437057	0.00374880	0.00320274	0.00272594	0.00231188

§ 382 (b). TABLE V—PART 2

SINKING FUND OR ANNUITY WHICH, INVESTED AT THE END OF EACH PERIOD, WILL AMOUNT TO $1

Periods	2¾%	3%	3½%	4%	4½%	5%	6%
1	1.	1.	1.	1.	1.	1.	1.
2	0.49321825	0.49261084	0.49140049	0.49019608	0.48899756	0.48780488	0.48543689
3	0.32433243	0.32353036	0.32193418	0.32034854	0.31877336	0.31720856	0.31410981
4	0.23992059	0.23902705	0.23725114	0.23549005	0.23374365	0.23201183	0.22859149
5	0.18929832	0.18835457	0.18648137	0.18462711	0.18279164	0.18097480	0.17739640
6	0.15557083	0.15459750	0.15266821	0.15076190	0.14887839	0.14701747	0.14336263
7	0.13149747	0.13050635	0.12854449	0.12660961	0.12470147	0.12281982	0.11913502
8	0.11345795	0.11245639	0.11047665	0.10852783	0.10660965	0.10472181	0.10103594
9	0.09944095	0.09843386	0.09644601	0.09449299	0.09257447	0.09069008	0.08702224
10	0.08823972	0.08723051	0.08524137	0.08329094	0.08137882	0.07950458	0.07586796
11	0.07908629	0.07807745	0.07609197	0.07414904	0.07224818	0.07038889	0.06679294
12	0.07146871	0.07046209	0.06848395	0.06655217	0.06466619	0.06282541	0.05927703
13	0.06503252	0.06402954	0.06206157	0.06014373	0.05827535	0.05645577	0.05296011
14	0.05952457	0.05852634	0.05657073	0.05466897	0.05282032	0.05102397	0.04758491
15	0.05475917	0.05376658	0.05182507	0.04994110	0.04811381	0.04634229	0.04296276
16	0.05059710	0.04961085	0.04768483	0.04582000	0.04401537	0.04226991	0.03895214
17	0.04693186	0.04595253	0.04404313	0.04219852	0.04041758	0.03869914	0.03544480
18	0.04368063	0.04270870	0.04081684	0.03899333	0.03723690	0.03554622	0.03235654
19	0.04077802	0.03981388	0.03794033	0.03613862	0.03440734	0.03274501	0.02962086
20	0.03817173	0.03721571	0.03536108	0.03358175	0.03187614	0.03024259	0.02718456
21	0.03581941	0.03487178	0.03303659	0.03128011	0.02960057	0.02799611	0.02500455
22	0.03368640	0.03274739	0.03093207	0.02919881	0.02754565	0.02597051	0.02304557
23	0.03174410	0.03081390	0.02901880	0.02730906	0.02568249	0.02413682	0.02127848
24	0.02996863	0.02904742	0.02727283	0.02558683	0.02398703	0.02247090	0.01967900
25	0.02833997	0.02742787	0.02567404	0.02401196	0.02243903	0.02095246	0.01822672
26	0.02684116	0.02593829	0.02420540	0.02256738	0.02102137	0.01956432	0.01690435
27	0.02545776	0.02456421	0.02285241	0.02123854	0.01971946	0.01829186	0.01569717
28	0.02417738	0.02329323	0.02160265	0.02001298	0.01852081	0.01712253	0.01459255
29	0.02298935	0.02211467	0.02044538	0.01887993	0.01741461	0.01604551	0.01357961

30	0.02188442	0.02101926	0.01937133	0.01783010	0.01639154	0.01505144	0.01264891
31	0.02085453	0.01999893	0.01837240	0.01685535	0.01544345	0.01413212	0.01179222
32	0.01989263	0.01904662	0.01744150	0.01594859	0.01456320	0.01328042	0.01100234
33	0.01899253	0.01815612	0.01657242	0.01510357	0.01374453	0.01249004	0.01027293
34	0.01814875	0.01732196	0.01575966	0.01431477	0.01298191	0.01175545	0.00959843
35	0.01735645	0.01653929	0.01499835	0.01357732	0.01227045	0.01107171	0.00897386
36	0.01661132	0.01580379	0.01428416	0.01288688	0.01160578	0.01043446	0.00839483
37	0.01590953	0.01511162	0.01361325	0.01223957	0.01098402	0.00983979	0.00785743
38	0.01524764	0.01445934	0.01298214	0.01163192	0.01040169	0.00928423	0.00735812
39	0.01462256	0.01384385	0.01238775	0.01106083	0.00985567	0.00876462	0.00689377
40	0.01403151	0.01326238	0.01182728	0.01052349	0.00934315	0.00827816	0.00646154
41	0.01347200	0.01271241	0.01129822	0.01001738	0.00886158	0.00782229	0.00605886
42	0.01294175	0.01219167	0.01079828	0.00954020	0.00840868	0.00739471	0.00568342
43	0.01243871	0.01169811	0.01032539	0.00908989	0.00798235	0.00699333	0.00533312
44	0.01196100	0.01122985	0.00987768	0.00866454	0.00758071	0.00661625	0.00500606
45	0.01150693	0.01078518	0.00945343	0.00826246	0.00720202	0.00626173	0.00470050
46	0.01107493	0.01036254	0.00905108	0.00788205	0.00684471	0.00592820	0.00441485
47	0.01066358	0.00996051	0.00866916	0.00752189	0.00650734	0.00561421	0.00414768
48	0.01027158	0.00957777	0.00830646	0.00718065	0.00618858	0.00531843	0.00389766
49	0.00989773	0.00921314	0.00796167	0.00685712	0.00588722	0.00503965	0.00366356
50	0.00954092	0.00886550	0.00763371	0.00655020	0.00560215	0.00477674	0.00344429
55	0.00797953	0.00734907	0.00621323	0.00523124	0.00438754	0.00366686	0.00253696
60	0.00672002	0.00613296	0.00508862	0.00420185	0.00345426	0.00282818	0.00187572
65	0.00569120	0.00514581	0.00418826	0.00339019	0.00273047	0.00218915	0.00139066
70	0.00484218	0.00433663	0.00346095	0.00274506	0.00216511	0.00169915	0.00103313
75	0.00413560	0.00366796	0.00286919	0.00222900	0.00172104	0.00132161	0.00076867
80	0.00354342	0.00311175	0.00238489	0.00181408	0.00137069	0.00102962	0.00057254
85	0.00304420	0.00264650	0.00198662	0.00147909	0.00109334	0.00080316	0.00042681
90	0.00262125	0.00225556	0.00165781	0.00120775	0.00087316	0.00062711	0.00031836
95	0.00226141	0.00192577	0.00138546	0.00098738	0.00069799	0.00049003	0.00023758
100	0.00195418	0.00164667	0.00115927	0.00080800	0.00055839	0.00038314	0.00017736

§ 383.

Table VI
Reciprocals and Square Roots

Ratio $(1+i)$	Reciprocal (Discount Multiplier)	Square Root (Quarterly Ratio)	Ratio $(1+i)$	Reciprocal (Discount Multiplier)	Square Root (Quarterly Ratio)
1.005	.99502488	1.00249688	1.02	.98039216	1.00995049
1.00525	.99477742	1.00262156	1.02025	.98015192	1.01007425
1.0055	.99453008	1.00274623	1.0205	.97991181	1.01019800
1.00575	.99428287	1.00287088	1.02075	.97967181	1.01032173
1.006	.99403579	1.00299551	1.021	.97943193	1.01044545
1.00625	.99378882	1.00312013	1.02125	.97919217	1.01056915
1.0065	.99354198	1.00324474	1.0215	.97895252	1.01069283
1.00675	.99329526	1.00336932	1.02175	.97871299	1.01081650
1.007	.99304866	1.00349390	1.022	.97847358	1.01094016
1.00725	.99280218	1.00361845	1.02225	.97823429	1.01106380
1.0075	.99255583	1.00374299	1.0225	.97799511	1.01118742
1.00775	.99230960	1.00386752	1.02275	.97775605	1.01131103
1.008	.99206349	1.00399203	1.023	.97751711	1.01143462
1.00825	.99181751	1.00411653	1.02325	.97727828	1.01155820
1.0085	.99157164	1.00424101	1.0235	.97703957	1.01168177
1.00875	.99132590	1.00436547	1.02375	.97680098	1.01180532
1.009	.99108028	1.00448992	1.024	.97656250	1.01192885
1.00925	.99083478	1.00461435	1.02425	.97632414	1.01205237
1.0095	.99058940	1.00473877	1.0245	.97608590	1.01217588
1.00975	.99034414	1.00486317	1.02475	.97584777	1.01229936
1.01	.99009901	1.00498756	1.025	.97560976	1.01242284
1.01025	.98985400	1.00511193	1.02525	.97537186	1.01254630
1.0105	.98960910	1.00523629	1.0255	.97513408	1.01266974
1.01075	.98936433	1.00536063	1.02575	.97489642	1.01279317
1.011	.98911968	1.00548496	1.026	.97465887	1.01291657
1.01125	.98887515	1.00560927	1.02625	.97442144	1.01303998
1.0115	.98863075	1.00573356	1.0265	.97418412	1.01316336
1.01175	.98838646	1.00585784	1.02675	.97394692	1.01328673
1.012	.98814229	1.00598211	1.027	.97370983	1.01341008
1.01225	.98789825	1.00610636	1.02725	.97347286	1.01353342
1.0125	.98765432	1.00623059	1.0275	.97323601	1.01365675
1.01275	.98741052	1.00635481	1.02775	.97299927	1.01378006
1.013	.98716683	1.00647901	1.028	.97276265	1.01390335
1.01325	.98692327	1.00660320	1.02825	.97252614	1.01402663
1.0135	.98667982	1.00672737	1.0285	.97228974	1.01414989

RECIPROCALS AND SQUARE ROOTS—*(Concluded)*

Ratio (1 + i)	Reciprocal (Discount Multiplier)	Square Root (Quarterly Ratio)	Ratio (1 + i)	Reciprocal (Discount Multiplier)	Square Root (Quarterly Ratio)
1.01375	.98643650	1.00685153	1.02875	.97205346	1.01427314
1.014	.98619329	1.00697567	1.029	.97181730	1.01439637
1.01425	.98595021	1.00709980	1.02925	.97158125	1.01451959
1.0145	.98570725	1.00722391	1.0295	.97134531	1.01464279
1.01475	.98546440	1.00734800	1.02975	.97110949	1.01476598
1.015	.98522167	1.00747208	1.03	.97087379	1.01488916
1.01525	.98497907	1.00759615	1.0325	.96852300	1.01612007
1.0155	.98473658	1.00772020	1.035	.96618357	1.01734950
1.01575	.98449422	1.00784423	1.0375	.96385542	1.01857744
1.016	.98425197	1.00796825	1.04	.96153846	1.01980390
1.01625	.98400984	1.00809226	1.0425	.95923261	1.02102889
1.0165	.98376783	1.00821625	1.045	.95693780	1.02225242
1.01675	.98352594	1.00834022	1.0475	.95465394	1.02347447
1.017	.98328417	1.00846418	1.05	.95238095	1.02469508
1.01725	.98304252	1.00858812	1.0525	.95011876	1.02591423
1.0175	.98280098	1.00871205	1.055	.94786730	1.02713193
1.01775	.98255957	1.00883596	1.0575	.94562648	1.02834819
1.018	.98231827	1.00895986	1.06	.94339623	1.02956301
1.01825	.98207709	1.00908374	1.0625	.94117647	1.03077641
1.0185	.98183603	1.00920761	1.065	.93896714	1.03198837
1.01875	.98159509	1.00933146	1.0675	.93676815	1.03319892
1.019	.98135427	1.00945530	1.07	.93457944	1.03440804
1.01925	.98111356	1.00957912	1.08	.92592593	1.03923048
1,0195	.98087298	1.00970293	1.09	.91743119	1.04403065
1.01975	.98063251	1.00982672	1.10	.90909091	1.04880885

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(References are to sections unless otherwise noted.)

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